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Remarks on common hypercyclic vectors

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Abstract

We treat the question of existence of common hypercyclic vectors for families of continuous linear operators. It is shown that for any continuous linear operator T on a complex Fréchet space X and a set $\Lambda \subseteq \mathbb{R}_+ \times \mathbb{C}$ which is not of zero three-dimensional Lebesgue measure, the family $\{aT + bI\colon (a,b) \in \Lambda\}$ has no common hypercyclic vectors. This allows to answer negatively questions raised by Godefroy and Shapiro and by Aron. We also prove a sufficient condition for a family of scalar multiples of a given operator on a complex Fréchet space to have a common hypercyclic vector. It allows to show that if $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ and $\varphi \in \mathcal{H}^\infty(\mathbb{D})$ is non-constant, then the family $\{zM_\varphi^\star\colon b^{-1} < |z| < a^{-1}\}$ has a common hypercyclic vector, where $M_\varphi : \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$, $M_\varphi f = \varphi f$, $a = \inf\{|\varphi(z)|\colon z \in \mathbb{D}\}$ and $b = \sup\{|\varphi(z)|\colon |z| \in \mathbb{D}\}$, providing an affirmative answer to a question by Bayart and Grivaux. Finally, extending a result of Costakis and Sambarino, we prove that the family $\{aT_b\colon a, b \in \mathbb{C} \setminus \{0\}\}$ has a common hypercyclic vector, where $T_b f(z) = f(z-b)$ acts on the Fréchet space $\mathcal{H}(\mathbb{C})$ of entire functions on one complex variable.

Keywords: Hypercyclic operators; Hypercyclic vectors

1. Introduction

All vector spaces in this article are assumed to be over $\mathbb{K}$ being either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. Throughout this paper all topological spaces and topological vector spaces **are assumed to be Hausdorff**. As usual, $\mathbb{Z}_+$ is the set of non-negative integers, $\mathbb{R}_+$ is the set of non-negative real numbers, $\mathbb{N}$ is the set of positive integers, $\mathbb{K}^\star = \mathbb{K} \setminus \{0\}$,

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$\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ and $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$. By a *compact interval* of the real line we mean a set of the shape $[a, b]$ with $-\infty < a < b < \infty$. That is, a singleton is **not** considered to be an interval. For topological vector spaces X and Y, $L(X, Y)$ stands for the space of continuous linear operators from X to Y. We write $L(X)$ instead of $L(X, X)$ and X^* instead of $L(X, \mathbb{K})$. For $T \in L(X, Y)$, the dual operator $T^* : Y^* \to X^*$ acts according to the formula $T^* f(x) = f(Tx)$. Recall [21] that an $\mathcal{F}$-space is a complete metrizable topological vector space and a Fréchet space is a locally convex $\mathcal{F}$-space. For a subset A of a vector space X, symbol $\mathtt{span}(A)$ stands for the linear span of A.

Definition 1.1. Let X and Y be topological spaces and $\mathcal{F} = \{T_a\colon a \in A\}$ be a family of continuous maps from X to Y. An element $x \in X$ is called *universal* for $\mathcal{F}$ if the orbit $\{T_a x\colon a \in A\}$ is dense in Y and $\mathcal{F}$ is said to be *universal* if it has a universal element. We denote the set of universal elements for $\mathcal{F}$ by the symbol $\mathfrak{U}(\mathcal{F})$. A continuous linear operator T acting on a topological vector space X is called *hypercyclic* if the family of its powers $\{T^n\colon n \in \mathbb{Z}_+\}$ is universal. Corresponding universal elements are called *hypercyclic vectors* for T. The set of hypercyclic vectors for T is denoted by $\mathfrak{H}(T)$. That is, $\mathfrak{H}(T) = \mathfrak{U}(\{T^n\colon n \in \mathbb{Z}_+\})$. If $\{T_a\colon a \in A\}$ is a family of continuous linear operators on topological vector space X, we denote

$$\mathfrak{H}\{T_a\colon a \in A\} = \bigcap_{a \in A} \mathfrak{H}(T_a).$$

That is, $\mathfrak{H}\{T_a\colon a \in A\}$ consists of all vectors $x \in X$ that are hypercyclic for each T_a, $a \in A$.

Recall that a topological space X is called *Baire* if the intersection of any countable family of dense open subsets of X is dense. Hypercyclic operators and universal families have been intensely studied during last few decades, see surveys [14,15] and references therein. It is well known [14] that the set of hypercyclic vectors of a hypercyclic operator on a separable metrizable Baire topological vector space is a dense G_δ-set. It immediately follows that any countable family of hypercyclic operators on such a space has a dense G_δ-set of common hypercyclic vectors (= hypercyclic for each member of the family). We are interested in the existence of common hypercyclic vectors for uncountable families of continuous linear operators. First results in this direction were obtained by Abakumov and Gordon [1] and León-Saavedra and Müller [18].

Theorem AG. *Let T be the backward shift on ℓ_2. That is, $T \in L(\ell_2)$, $Te_0 = 0$ and $Te_n = e_{n-1}$ for $n \in \mathbb{N}$, where $\{e_n\}_{n \in \mathbb{Z}_+}$ is the standard orthonormal basis of ℓ_2. Then $\mathfrak{H}\{aT\colon a \in \mathbb{K},\ |a| > 1\}$ is a dense G_δ-set.*

The following result is of completely different flavor. It is proven in [18] for continuous linear operators on Banach spaces although the proof can be easily adapted [23] for continuous linear operators acting on arbitrary topological vector spaces.

Theorem LM. *Let X be a complex topological vector space and $T \in L(X)$. Then $\mathfrak{U}(\mathcal{F}) = \mathfrak{H}(zT) = \mathfrak{H}(T)$ for any $z \in \mathbb{T}$, where $\mathcal{F} = \{wT^n\colon w \in \mathbb{T},\ n \in \mathbb{Z}_+\}$. In particular, $\mathfrak{H}\{zT\colon z \in \mathbb{T}\} = \mathfrak{H}(T)$.*

It follows that the family $\{zT\colon z \in \mathbb{T}\}$ has a common hypercyclic vector, whenever T is a hypercyclic operator. A result similar to the above one was recently obtained by Conejero, Müller and Peris [9] for operators acting on separable $\mathcal{F}$-spaces (see [23] for a proof in a more general setting). Recall that a family $\{T_t\}_{t\in\mathbb{R}_+}$ of continuous linear operators on a topological vector space is called an *operator semigroup* if $T_0 = I$ and $T_{t+s} = T_tT_s$ for any $t, s \in \mathbb{R}_+$.

Theorem CMP. *Let X be a topological vector space and $\{T_t\}_{t\in\mathbb{R}_+}$ be an operator semigroup on X. Assume also that the map $(t, x) \mapsto T_tx$ from $\mathbb{R}_+ \times X$ to X is continuous. Then $\mathfrak{H}(T_t) = \mathfrak{U}(\mathcal{F})$ for any $t > 0$, where $\mathcal{F} = \{T_s\colon s > 0\}$. In particular, $\mathfrak{H}\{T_s\colon s > 0\} = \mathfrak{H}(T_t)$ for any $t > 0$.*

It follows that if $\{T_t\}_{t\in\mathbb{R}_+}$ is an operator semigroup such that the map $(t, x) \mapsto T_tx$ is continuous and there exists $t > 0$ for which T_t is hypercyclic, then the family $\{T_s\colon s > 0\}$ has a common hypercyclic vector. Bayart [2] provided families of composition operators on the space of holomorphic functions on $\mathbb{D}$, which have common hypercyclic vectors. Costakis and Sambarino [11], Bayart and Matheron [4], Chan and Sanders [8] and Gallardo-Gutiérrez and Partington [12] proved certain sufficient conditions for a set of families of continuous linear operators to have a common universal vector. In all the mentioned papers the criteria were applied to specific sets of families. For instance, Costakis and Sambarino [11] proved the following theorem.

Theorem CS. *Let $\mathcal{H}(\mathbb{C})$ be the complex Fréchet space of entire functions on one variable, $D \in L(\mathcal{H}(\mathbb{C}))$ be the differentiation operator $Df = f'$ and for each $a \in \mathbb{C}$, $T_a \in L(\mathcal{H}(\mathbb{C}))$ be the translation operator $T_af(z) = f(z - a)$. Then $\mathfrak{H}\{T_a\colon a \in \mathbb{C}^\star\}$, $\mathfrak{H}\{aT_1\colon a \in \mathbb{C}^\star\}$ and $\mathfrak{H}\{aD\colon a \in \mathbb{C}^\star\}$ are dense G_δ-sets.*

The criteria by Bayart and Matheron were applied to various families of operators including families of weighted translations on $L^p(\mathbb{R})$, composition operators on Hardy spaces $\mathcal{H}^p(\mathbb{D})$ and backward weighted shifts on ℓ_p. We would like to mention just one example of the application of the criterion from [4], which is related to our results.

Example BM. As in Theorem CS, let T_a be translation operators on $\mathcal{H}(\mathbb{C})$. For each $s \in \mathbb{R}_+$ and $z \in \mathbb{T}$, consider the family $\mathcal{F}_{s,z} = \{n^sT_{nz}\colon n \in \mathbb{Z}_+\}$. Then

$$\bigcap_{(s,z)\in\mathbb{R}_+\times\mathbb{T}} \mathfrak{U}(\mathcal{F}_{s,z}) \text{ is a dense } G_\delta\text{-subset of } \mathcal{H}(\mathbb{C}).$$

Chan and Sanders [8] found common universal elements of certain sets of families of backward weighted shifts on ℓ_2. Gallardo-Gutiérrez and Partington [12] proved a modification of the Costakis–Sambarino criterion and applied it to obtain common hypercyclic vectors for families of adjoint multipliers and composition operators on Hardy spaces. Finally, we would like to mention the following application by Costakis and Mavroudis [10] of the Bayart–Matheron criterion.

Theorem CM. *Let D be the differentiation operator on $\mathcal{H}(\mathbb{C})$ and p be a non-constant polynomial. Then $\mathfrak{H}\{ap(D)\colon a \in \mathbb{C}^\star\}$ is a dense G_δ-set.*

Although the most of the mentioned criteria look quite general, they are basically not applicable to finding common hypercyclic vectors of families that are not smoothly labeled by **one** real parameter. Note that although the families in Theorems AG, CS and CM are formally speaking labeled by a complex parameter a, Theorem LM allows to reduce them to families labeled by one real parameter. Example BM is, of course, genuinely two-parametric, but it is not about a common hypercyclic vector. On the other hand, one can artificially produce huge families of operators with a common hypercyclic vector. For example, take all operators for which a given vector is hypercyclic. The following result provides a common hypercyclic vector for a two-parametric family of operators. It strengthens the first part of Theorem CS and kind of improves Example BM.

Theorem 1.2. *Let T_a for $a \in \mathbb{C}$ be the translation operator $T_a f(z) = f(z-a)$ acting on the complex Fréchet space $\mathcal{H}(\mathbb{C})$ of entire functions on one complex variable. Then $\mathfrak{H}\{bT_a\colon a,b \in \mathbb{C}^\star\}$ is a dense G_δ-set.*

A common hypercyclic vector from the above theorem is even more monstrous than the holomorphic monsters provided by Theorem CS. Godefroy and Shapiro [13] considered adjoint multiplication operators on function Hilbert spaces. Recall that if U is a connected open subset of $\mathbb{C}^m$, then a *function Hilbert space* $\mathcal{H}$ on U is a Hilbert space consisting of functions $f : U \to \mathbb{C}$ holomorphic on U such that for any $z \in U$ the evaluation functional $\chi_z : \mathcal{H} \to \mathbb{C}$, $\chi_z(f) = f(z)$ is continuous. A *multiplier* for $\mathcal{H}$ is a function $\varphi : U \to \mathbb{C}$ such that $\varphi f \in \mathcal{H}$ for each $f \in \mathcal{H}$. It is well known [13] that any multiplier is bounded and holomorphic. Each multiplier gives rise to the multiplication operator $M_\varphi \in L(\mathcal{H})$, $M_\varphi f = \varphi f$ (continuity of M_φ follows from the Banach closed graph theorem). Its Hilbert space adjoint $M_\varphi^\star$ is called an *adjoint multiplication operator*. Godefroy and Shapiro proved that there is $f \in \mathcal{H}$, which is cyclic for $M_\varphi^\star$ for any non-constant multiplier φ for $\mathcal{H}$ and demonstrated that if $\varphi : U \to \mathbb{C}$ is a non-constant multiplier for $\mathcal{H}$ and $\varphi(U) \cap \mathbb{T} \neq \varnothing$, then $M_\varphi^\star$ is hypercyclic, see also the related paper by Bourdon and Shapiro [7]. Godefroy and Shapiro also raised the following question [13, p. 263].

Question GS. *Let $\mathcal{H}$ be a Hilbert function space on a connected open subset U of $\mathbb{C}^m$. Does the family of all hypercyclic adjoint multiplications on $\mathcal{H}$ have a common hypercyclic vector?*

Recall that any $T \in L(\mathcal{H}(\mathbb{C}))$ such that T is not a scalar multiple of the identity and $TD = DT$ is hypercyclic. The following question was raised by Richard Aron.

Question A. *Let $\mathcal{D}$ be the family of all continuous linear operators on $\mathcal{H}(\mathbb{C})$, which are not scalar multiples of the identity and which commute with the differentiation operator D. Is it true that there is a common hypercyclic vector for all operators from the family $\mathcal{D}$?*

The next result allows us to answer negatively both of the above questions.

Theorem 1.3. *Let X be a complex topological vector space such that $X^* \neq \{0\}$, $T \in L(X)$ and Λ be a subset of $\mathbb{R}_+ \times \mathbb{C}$. Assume also that the family $\{aT + bI\colon (a,b) \in \Lambda\}$ has a common hypercyclic vector. Then the set Λ has zero three-dimensional Lebesgue measure.*

Corollary 1.4. *The family $\{aD + bI\colon a > 0,\ b \in \mathbb{C}\}$ of continuous linear operators on $\mathcal{H}(\mathbb{C})$ does not have a common hypercyclic vector.*

Corollary 1.5. *Let $\mathcal{H}$ be a Hilbert function space on a connected open subset U of $\mathbb{C}^m$ and φ be a non-constant multiplier for $\mathcal{H}$. Then the family $\{M^\star_{\bar{b}+a\varphi}\colon a > 0,\ b \in \mathbb{C},\ (\bar{b}+a\varphi)(U) \cap \mathbb{T} \neq \varnothing\}$ of hypercyclic operators does not have a common hypercyclic vector.*

Corollaries 1.4 and 1.5 follow from Theorem 1.3 because $M^\star_{\bar b+a\varphi}=aM^\star_\varphi+bI$ and the sets of pairs (a,b) involved in the definition of the families in Corollaries 1.4 and 1.5 are non-empty open subsets of $\mathbb{R}_+\times\mathbb{C}$ and therefore have non-zero 3-dimensional Lebesgue measure. In fact, Theorem 1.3 shows that even relatively small subfamilies of the families from Questions GS and A fail to have common hypercyclic vectors. As usual, $\mathcal{H}^2(\mathbb{D})$ is the Hardy space of the unit disk. It is well known that $\mathcal{H}^2(\mathbb{D})$ is a function Hilbert space on $\mathbb{D}$ and the set of multipliers for $\mathcal{H}^2(\mathbb{D})$ is the space $\mathcal{H}^\infty(\mathbb{D})$ of bounded holomorphic functions $f:\mathbb{D}\to\mathbb{C}$. Let $\varphi\in\mathcal{H}^\infty(\mathbb{D})$ be non-constant. Using the mentioned criterion by Godefroy and Shapiro for hypercyclicity of adjoint multiplications together with the fact that a contraction or its inverse cannot be hypercyclic, we see that $zM^\star_\varphi=M^\star_{\bar z\varphi}$ is hypercyclic if and only if $b^{-1}<|z|<a^{-1}$, where $a=\inf_{z\in\mathbb{D}}|\varphi(z)|$ and $b=\sup_{z\in\mathbb{D}}|\varphi(z)|$. Probably, expecting the answer to Question GS to be negative, Bayart and Grivaux [3] raised the following question.

Question BG. *Let $\varphi\in\mathcal{H}^\infty(\mathbb{D})$ be non-constant, $a=\inf_{z\in\mathbb{D}}|\varphi(z)|$ and $b=\sup_{z\in\mathbb{D}}|\varphi(z)|$. Is it true that the family $\{zM^\star_\varphi\colon\ b^{-1}<|z|<a^{-1}\}$ has common hypercyclic vectors?*

We prove a sufficient condition on a family of scalar multiples of a given operator to have a common hypercyclic vector and use it to answer Question BG affirmatively. It is worth noting that Gallardo-Gutiérrez and Partington [12] found a partial affirmative answer to the above question.

Theorem 1.6. *Let X be a separable complex $\mathcal{F}$-space, $T\in L(X)$ and $0\leqslant a<b\leqslant\infty$. Assume also that there is a map $(k,c)\mapsto F_{k,c}$ sending a pair $(k,c)\in\mathbb{N}\times(a,b)$ to a subset $F_{k,c}$ of X satisfying the following properties*:

(1.6.1) $F_{k,c}\subseteq\bigcup_{w\in\mathbb{T}}\ker(T^k-wc^kI)$ *for each* $(k,c)\in\mathbb{N}\times(a,b)$;
(1.6.2) $\{c\in(a,b)\colon\ F_{k,c}\cap V\neq\varnothing\}$ *is open in* (a,b) *for any open subset* V *of* X *and* $k\in\mathbb{N}$;
(1.6.3) $F_c=\bigcup_{k=1}^\infty F_{k,c}$ *is dense in* X *for any* $c\in(a,b)$;
(1.6.4) *For any* $k_1,\dots,k_n\in\mathbb{N}$, *there is* $k\in\mathbb{N}$ *such that* $\bigcup_{j=1}^n F_{k_j,c}\subseteq F_{k,c}$ *for each* $c\in(a,b)$.

Then $\mathfrak{H}\{zT\colon\ b^{-1}<|z|<a^{-1}\}$ is a dense G_δ-set.

Note that (1.6.1) is satisfied if $F_{k,c}\subseteq\ker(T^k-c^kI)$, which is the case in all following applications of Theorem 1.6. If X is a complex locally convex topological vector space and U is a non-empty open subset of $\mathbb{C}^m$, then we say that $f:U\to X$ is *holomorphic* if f is continuous and for each $g\in X^*$, $g\circ f:U\to\mathbb{C}$ is holomorphic.

Theorem 1.7. *Let $m\in\mathbb{N}$, X be a complex Fréchet space, $T\in L(X)$ and U be a connected open subset of $\mathbb{C}^m$. Assume also that there exist holomorphic maps $f:U\to X$ and $\varphi:U\to\mathbb{C}$ such that φ is non-constant, $Tf(z)=\varphi(z)f(z)$ for each $z\in U$ and $\mathrm{span}\{f(z)\colon\ z\in U\}$ is dense in X. Denote $a=\inf_{z\in U}|\varphi(z)|$ and $b=\sup_{z\in U}|\varphi(z)|$. Then $\mathfrak{H}\{zT\colon\ b^{-1}<|z|<a^{-1}\}$ is a dense G_δ-set.*

Corollary 1.8. *Let $m\in\mathbb{N}$, U be connected non-empty open subset of $\mathbb{C}^m$, $\mathcal{H}$ be a function Hilbert space on U and φ be a non-constant multiplier for $\mathcal{H}$, $a=\inf_{z\in U}|\varphi(z)|$ and $b=\sup_{z\in U}|\varphi(z)|$. Then $\mathfrak{H}\{zT\colon\ b^{-1}<|z|<a^{-1}\}$ is a dense G_δ-set.*

Corollary 1.9. *Let $T \in L(\mathcal{H}(\mathbb{C}))$ be such that $TD = DT$ and $T \neq cI$ for any $c \in \mathbb{C}$. Then $\mathfrak{H}\{zT\colon z \in \mathbb{C}^\star\}$ is a dense G_δ-set.*

Corollary 1.10. *Let X be a separable Fréchet space, $T \in L(X)$, $0 \leqslant a < b \leqslant \infty$ and $T \in L(X)$. Assume also that for any $\alpha, \beta \in \mathbb{R}$ such that $a < \alpha < \beta < b$, there exist a dense subset E of X and a map $S : E \to E$ such that $TSx = x$, $\alpha^{-n}T^n x \to 0$ and $\beta^n S^n x \to 0$ for each $x \in E$. Then $\mathfrak{H}\{zT\colon b^{-1} < |z| < a^{-1}\}$ is a dense G_δ-set.*

Note that Corollary 1.8 gives an affirmative answer to Question BG, Corollary 1.9 contains Theorem CM as a particular case, while Corollary 1.10 may be considered as an analog of the Kitai criterion [17]. The above results on common hypercyclic vectors for scalar multiples of a given operator may lead to an impression that for $0 < a < b < \infty$ and a continuous linear operator T on a Fréchet space, hypercyclicity of aT and bT implies the existence of common hypercyclic vectors for the family $\{cT\colon a \leqslant c \leqslant b\}$. This impression is utterly false as follows from the next theorem. For a continuous linear operator T on a topological vector space X, we denote

$$M_T = \{c > 0\colon\ cT \text{ is hypercyclic}\}.$$

Theorem 1.11.

I. *There exists $S \in L(\ell_2)$ such that $M_S = \{1, 2\}$.*
II. *There exists $T \in L(\ell_2)$ such that M_T is an open interval, but any $A \subset \mathbb{R}_+$ for which the family $\{cT\colon c \in A\}$ has common hypercyclic vectors is of zero Lebesgue measure.*

2. Yet another general criterion

Lemma 2.1. *Let A be a set and X, Y and Ω be topological spaces such that Ω is compact. For each $a \in A$ let $(\omega, x) \mapsto F_{a,\omega}x$ be a continuous map from $\Omega \times X$ to Y. For any $\omega \in \Omega$ let $\mathcal{F}_\omega = \{F_{a,\omega}\colon a \in A\}$ treated as a family of continuous maps from X to Y. Denote $\mathfrak{U}^* = \bigcap_{\omega\in\Omega} \mathfrak{U}(\mathcal{F}_\omega)$. Then*

$$G_V = \bigcap_{\omega\in\Omega} \bigcup_{a\in A} F_{a,\omega}^{-1}(V) \text{ is open in } X \text{ for any open subset } V \text{ of } Y. \tag{2.1}$$

Moreover, for any base $\mathcal{V}$ of topology of Y,

$$\mathfrak{U}^* = \bigcap_{V\in\mathcal{V}} G_V. \tag{2.2}$$

In particular, $\mathfrak{U}^*$ is a G_δ-set if Y is second countable.

Proof. Let $x \in G_V$. Then for any $\omega \in \Omega$, there exists $a(\omega) \in A$ such that $F_{a(\omega),\omega}x \in V$. Continuity of the map $\omega \mapsto F_{a,\omega}x$ implies that for each $\omega \in \Omega$, $W_\omega = \{\alpha \in \Omega\colon F_{a(\omega),\alpha}x \in V\}$ is an open neighborhood of ω in Ω. Since any Hausdorff compact space is regular, for any $\omega \in \Omega$, we can pick an open neighborhood W'_ω of ω in Ω such that, $\overline{W'_\omega} \subseteq W_\omega$. Since $\{W'_\omega\colon \omega \in \Omega\}$ is an open covering of the compact space Ω, there are $\omega_1, \dots, \omega_n \in \Omega$ such that $\Omega = \bigcup_{j=1}^n W'_{\omega_j}$. Continuity of the map $(\alpha, z) \mapsto F_{a,\alpha}z$ and compactness of $\overline{W'_\omega}$ imply that for any $j \in \{1, \dots, n\}$, there is a neighborhood U_j of x in X such that $F_{a(\omega_j),\alpha}z \in V$ for any $\alpha \in \overline{W'_{\omega_j}}$ and $z \in U_j$. Let $U = \bigcap_{j=1}^{n} U_j$. Since $\Omega = \bigcup_{j=1}^{n} W'_{\omega_j}$, for any $z \in U$ and $\omega \in \Omega$, there exists $j \in \{1, \dots, n\}$ such that $F_{a(\omega_j),\omega}z \in V$. Hence $U \subseteq G_V$. Thus any point of G_V is interior and therefore G_V is open. The equality (2.2) follows immediately from the definition of $\mathfrak{U}^*$. $\square$

The main tool in the proof of Theorem 1.2 is the following criterion. It is a simultaneous generalization of results by Chan and Sanders [8, Theorem 2.1] and Grosse-Erdmann [14, Theorem 1]. The latter is exactly the next proposition in the case when Ω is a singleton.

Proposition 2.2. *Let A be a set and X, Y, Ω be topological spaces such that X is Baire, Y is second countable and Ω is compact. For each $a \in A$, let $(\omega, x) \mapsto F_{a,\omega}x$ be a continuous map from $\Omega \times X$ to Y. Let $\mathcal{F}_\omega = \{F_{a,\omega}\colon a \in A\}$ for $\omega \in \Omega$ and $\mathfrak{U}^* = \bigcap_{\omega\in\Omega} \mathfrak{U}(\mathcal{F}_\omega)$. Then $\mathfrak{U}^*$ is a G_δ-subset of X. Moreover, the following conditions are equivalent.*

(2.2.1) *$\mathfrak{U}^*$ is dense in X.*

(2.2.2) *For any non-empty open set U in X and any non-empty open set V in Y, there exists $x \in U$ such that $V \cap \{F_{a,\omega}x\colon a \in A\} \neq \varnothing$ for each $\omega \in \Omega$.*

Proof. Let $\mathcal{V}$ be a countable base of the topology of Y. By Lemma 2.1, $\mathfrak{U}^*$ is a G_δ-set. Assume that (2.2.2) is satisfied. For any $V \in \mathcal{V}$ and $n \in \mathbb{N}$, condition (2.2.2) implies that G_V defined by (2.1) is dense in X. By Lemma 2.1, each G_V is a dense open subset of X. Since X is Baire, (2.2) implies that $\mathfrak{U}^*$ is a dense G_δ-subset of X. Hence (2.2.2) implies (2.2.1). Next, assume that (2.2.1) is satisfied and U, V are non-empty open subsets of X and Y respectively. Since $\mathfrak{U}^*$ is dense in X, there is $x \in \mathfrak{U}^* \cap U$. Let $\omega \in \Omega$. Since $x \in \mathfrak{U}(\mathcal{F}_\omega)$, there is $a \in A$ such that $F_{a,\omega}x \in V$. Hence (2.2.2) is satisfied. $\square$

Using Proposition 2.2 and the fact that in a Baire topological space the class of dense G_δ sets is closed under countable intersections, we immediately obtain the following corollary.

Corollary 2.3. *Let A be a set and X, Y, Ω be topological spaces such that X is Baire, Y is second countable and Ω is the union of its compact subsets Ω_n for $n \in \mathbb{N}$. For each $a \in A$, let $(\omega, x) \mapsto F_{a,\omega}x$ be a continuous map from $\Omega \times X$ to Y. Let $\mathcal{F}_\omega = \{F_{a,\omega}\colon a \in A\}$ for $\omega \in \Omega$ and $\mathfrak{U}^* = \bigcap_{\omega\in\Omega} \mathfrak{U}(\mathcal{F}_\omega)$. Then $\mathfrak{U}^*$ is a G_δ-subset of X. Moreover, the following conditions are equivalent.*

(2.3.1) *$\mathfrak{U}^*$ is dense in X.*

(2.3.2) *For each $n \in \mathbb{N}$, any non-empty open set U in X and any non-empty open set V in Y, there exists $x \in U$ such that $V \cap \{F_{a,\omega}x\colon a \in A\} \neq \varnothing$ for each $\omega \in \Omega_n$.*

Recall that if X is a topological vector space, A is a set and $\{f_n\}_{n\in\mathbb{Z}_+}$ is a sequence of maps from A to X, then we say that f_n *uniformly converges to* 0 on A if for any neighborhood W of 0 in X, there is $n \in \mathbb{Z}_+$ such that $f_k(a) \in W$ for any $a \in A$ and any $k \geqslant n$.

Definition 2.4. Let X and Y be topological vector spaces, A be a set and Ω be a topological space. We use the symbol

$$\mathcal{L}_{\Omega,A}(X, Y)$$

to denote the set of maps $(\omega,a,n,x)\mapsto T_{\omega,a,n}x$ from $\Omega\times A\times\mathbb{Z}_+\times X$ to Y such that $T_{\omega,a,n}\in L(X,Y)$ for each $(\omega,a,n)\in\Omega\times A\times\mathbb{Z}_+$ and the map $(\omega,x)\mapsto T_{\omega,a,n}x$ from $\Omega\times X$ to X is continuous for any $(a,n)\in A\times\mathbb{Z}_+$. If $T\in\mathcal{L}_{\Omega,A}(X,Y)$ is fixed, $\Lambda\subseteq\mathbb{Z}_+$, $u\in X$ and U is a subset of Y, we denote

$$M(u,\Lambda,U)=\{\omega\in\Omega\colon\ T_{\omega,a,n}u\in U \text{ for some } n\in\Lambda \text{ and } a\in A\}. \tag{2.3}$$

Proposition 2.5. *Let A be a set, X be a Baire topological vector space, Y be a separable metrizable topological vector space, Ω be a compact topological space and $T\in\mathcal{L}_{\Omega,A}(X,Y)$ be such that*

(2.5.1) $E=\{x\in X\colon\ T_{\omega,a,n}x\to 0$ *as* $n\to\infty$ *uniformly on* $\Omega\times A\}$ *is dense in* X;
(2.5.2) *for any non-empty open subset U of Y, there exist $m\in\mathbb{N}$ and compact subsets $\Omega_1,\dots,\Omega_m$ of Ω such that $\Omega=\bigcup_{j=1}^m\Omega_j$ and for any $j\in\{1,\dots,m\}$, $l\in\mathbb{Z}_+$ and a neighborhood W of 0 in X, there are a finite set $\Lambda\subset\mathbb{Z}_+$ and $u\in W$ for which* $\min\Lambda\geqslant l$ *and $\Omega_j\subseteq M(u,\Lambda,U)$.*

Then $\mathfrak{U}^*=\bigcap_{\omega\in\Omega}\mathfrak{U}(\mathcal{F}_\omega)$ is a dense G_δ-subset of X, where $\mathcal{F}_\omega=\{T_{\omega,a,n}\colon\ a\in A,\ n\in\mathbb{Z}_+\}$.

Proof. Let U_0 be a non-empty open subset of X and U be a non-empty open subset of Y. Pick $y_0\in U$ and a neighborhood W of zero in Y such that $y_0+W+W\subseteq U$. Then $V=y_0+W$ is a non-empty open subset of Y and $V+W\subseteq U$. According to (2.5.2), there exist compact subsets $\Omega_1,\dots,\Omega_m$ of Ω such that $\Omega=\bigcup_{j=1}^m\Omega_j$ and

$$\begin{aligned}&\text{for any } j\in\{1,\dots,m\},\ l\in\mathbb{Z}_+ \text{ and any neighborhood } W_1 \text{ of } 0 \text{ in } X,\ \text{there are}\\ &\text{a finite set } \Lambda\subset\mathbb{Z}_+ \text{ and } u\in W_1 \text{ such that } \min\Lambda\geqslant l \text{ and } \Omega_j\subseteq M(u,\Lambda,V).\end{aligned} \tag{2.4}$$

We shall construct inductively $u_0,\dots,u_m\in E\cap U_0$ and finite sets $\Lambda_1,\dots,\Lambda_m\subset\mathbb{Z}_+$ such that for $0\leqslant j\leqslant m$,

$$\Omega_p\subseteq M(u_j,\Lambda_p,U)\quad \text{for } 1\leqslant p\leqslant j. \tag{2.5}$$

By (2.5.1), the linear space E is dense in X. Hence we can pick $u_0\in U_0\cap E$, which will serve as the basis of induction. Assume now that $1\leqslant q\leqslant m$ and $u_0,\dots,u_{q-1}\in E\cap U_0$ and finite subsets $\Lambda_1,\dots,\Lambda_{q-1}$ of $\mathbb{Z}_+$ satisfying (2.5) with $0\leqslant j\leqslant q-1$ are already constructed. We shall construct $u_q\in E\cap U_0$ and a finite subset Λ_q of $\mathbb{Z}_+$ satisfying (2.5) with $j=q$. Consider the set

$$G=\bigl\{u\in X\colon\ \Omega_p\subseteq M(u,\Lambda_p,U) \text{ for } 1\leqslant p\leqslant q-1\bigr\}.$$

Since Ω_p are compact and U is open, Lemma 2.1 implies that G is open in X. According to (2.5) with $j=q-1$, $u_{q-1}\in G$. Since $u_{q-1}\in E$, there exists $l\in\mathbb{Z}_+$ such that

$$T_{\omega,a,n}u_{q-1}\in W\quad \text{for any } n\geqslant l \text{ and any } (\omega,a)\in\Omega\times A. \tag{2.6}$$

Since $u_{q-1} \in G \cap U_0$, and $G \cap U_0$ is open in X, $W_1 = (G \cap U_0) - u_{q-1}$ is a neighborhood of 0 in X. According to (2.4), there exists a finite subset Λ_q of $\mathbb{Z}_+$ such that

$$\min \Lambda_q \geqslant l \quad \text{and} \quad G_1 = \{u \in W_1 \colon\ \Omega_q \subseteq M(u, \Lambda_q, V)\} \neq \emptyset.$$

By Lemma 2.1, G_1 is open in X. Since E is dense in X, we can pick $u \in G_1 \cap E$. Denote $u_q = u_{q-1} + u$. We shall see that u_q and Λ_q satisfy (2.5) with $j = q$.

Since $u_{q-1}, u \in E$ and E is a linear space, we have $u_q \in E$. Since $u \in W_1 = (G \cap U_0) - u_{q-1}$, we get $u_q \in G \cap U_0$. In particular, $u_q \in U_0 \cap E$ and $u_q \in G$. By definition of G, $\Omega_p \subseteq M(u_q, \Lambda_p, U)$ for $1 \leqslant p \leqslant q-1$. Since $u \in G_1$, for any $\omega \in \Omega_q$, there exist $n_\omega \in \Lambda_q$ and $a_\omega \in A$ such that $T_{\omega,a_\omega,n_\omega} u \in V$. Since $n_\omega \in \Lambda_q$ and $\min \Lambda_q \geqslant l$, we have $n_\omega \geqslant l$. According to (2.6), $T_{\omega,a_\omega,n_\omega} u_{q-1} \in W$. The equality $u_q = u_{q-1} + u$ and linearity of $T_{\omega,a_\omega,n_\omega}$ imply $T_{\omega,a_\omega,n_\omega} u_q \in V + W \subseteq U$. Since $\omega \in \Omega_q$ is arbitrary, $\Omega_q \subseteq M(u_q, \Lambda_q, U)$. This completes the proof of (2.5) for $j = q$ and the inductive construction of $u_0, \dots, u_m$ and $\Lambda_1, \dots, \Lambda_m$ satisfying (2.5).

Since Ω is the union of Ω_j with $1 \leqslant j \leqslant m$, (2.5) for $j = m$ implies that $u_m \in U_0$ and $\Omega = M(u_m, \mathbb{Z}_+, U)$. That is, for any $\omega \in \Omega$ there are $a \in A$ and $n \in \mathbb{Z}_+$ such that $T_{\omega,a,n} u_m \in U$. Since U_0 and U are arbitrary non-empty open subsets of X and Y respectively, condition (2.2.2) is satisfied. By Proposition 2.2, $\mathfrak{U}^*$ is a dense G_δ-subset of X. □

Since for any $\delta > 0$, any compact interval of the real line is the union of finitely many intervals of length $\leqslant \delta$, we immediately obtain the following corollary.

Corollary 2.6. *Let A be a set, X be a Baire topological vector space, Y be a separable metrizable topological vector space, Ω be a compact interval of $\mathbb{R}$ and $T \in \mathcal{L}_{\Omega,A}(X,Y)$ be such that* (2.5.1) *is satisfied and*

(2.6.2) *for any non-empty open subset U of Y, there exists $\delta > 0$ such that for any compact interval $J \subseteq \Omega$ of length $\leqslant \delta$, $l \in \mathbb{Z}_+$ and a neighborhood W of 0 in X, there exist a finite set $\Lambda \subset \mathbb{Z}_+$ and $u \in W$ for which $\min \Lambda \geqslant l$ and $J \subseteq M(u, \Lambda, U)$.*

Then $\mathfrak{U}^* = \bigcap_{\omega\in\Omega} \mathfrak{U}(\mathcal{F}_\omega)$ is a dense G_δ-subset of X, where $\mathcal{F}_\omega = \{T_{\omega,a,n}\colon\ a \in A,\ n \in \mathbb{Z}_+\}$.

3. Operator groups with the Runge property

In this section we prove a statement more general than of Theorem 1.2.

Definition 3.1. Let X be a locally convex topological vector space and $\{T_z\}_{z\in\mathbb{C}}$ be an operator group. That is, $T_z \in L(X)$ for each $z \in \mathbb{C}$, $T_0 = I$ and $T_{z+w} = T_z T_w$ for any $z, w \in \mathbb{C}$. We say that $\{T_z\}_{z\in\mathbb{C}}$ has the *Runge property* if for any continuous seminorm p on X there exists $c = c(p) > 0$ such that for any finite set S of complex numbers satisfying $|z - z'| \geqslant c$ for $z, z' \in S$, $z \neq z'$, any $\varepsilon > 0$ and $\{x_z\}_{z\in S} \in X^S$, there is $x \in X$ such that $p(T_z x - x_z) < \varepsilon$ for each $z \in S$.

Lemma 3.2. *For each $a \in \mathbb{C}$ let $T_a \in L(\mathcal{H}(\mathbb{C}))$ be the translation operator $Tf(z) = f(z-a)$. Then the group $\{T_a\}_{a\in\mathbb{C}}$ has the Runge property.*

Proof. Let p be a continuous seminorm on $\mathcal{H}(\mathbb{C})$. Then there exists $a > 0$ such that $p(f) \leqslant q(f)$ for each $f \in \mathcal{H}(\mathbb{C})$, where $q(f) = a \max_{|z|\leqslant a} |f(z)|$. Take any $c > 2a$. We shall show that c

satisfies the condition from Definition 3.1. Let $\varepsilon > 0$, S be a finite set of complex numbers such that $|z - z'| \geqslant c$ for $z, z' \in S$, $z \neq z'$ and $\{f_z\}_{z \in S} \in \mathcal{H}(\mathbb{C})^S$. For each $z \in S$ consider the disk $D_z = \{w \in \mathbb{C}\colon\ |z + w| \leqslant a\}$ and let $D = \bigcup_{z \in S} D_z$. Since $|z - z'| \geqslant c$ for $z, z' \in S$, $z \neq z'$, the closed disks D_z are pairwise disjoint. It follows that $\mathbb{C} \setminus D$ is connected. By the classical Runge theorem, any function holomorphic in a neighborhood of the compact set D can be with any prescribed accuracy uniformly on D approximated by a polynomial. Thus there is a polynomial f such that $\sup_{w \in D_z} |f(w) - f_z(z + w)| < \varepsilon/a$ for any $z \in S$. Equivalently, $\sup_{|w| \leqslant a} |f(w - z) - f_z(w)| < \delta$ for any $z \in S$. Using the definitions of T_z and q, we obtain $p(T_z f - f_z) \leqslant q(T_z f - f_z) < \varepsilon$ for each $z \in S$. $\square$

It is also easy to show that the translation group satisfies the Runge property when acting on the Fréchet space $C(\mathbb{C})$ of continuous functions $f : \mathbb{C} \to \mathbb{C}$ with the topology of uniform convergence on compact sets. Recall that an operator semigroup $\{T_t\}$ is called *strongly continuous* if the map $(t, x) \mapsto T_t x$ is separately continuous.

Theorem 3.3. *Let X be a separable Fréchet space and $\{T_z\}_{z \in \mathbb{C}}$ be a strongly continuous operator group on X with the Runge property. Then the family $\{aT_b\colon\ a \in \mathbb{K}^\star,\ b \in \mathbb{C}^\star\}$ has a dense G_δ-set of common hypercyclic vectors.*

According to Lemma 3.2, Theorem 1.2 is a particular case of Theorem 3.3. The rest of this section is devoted to the proof of Theorem 3.3. We need a couple of technical lemmas.

Lemma 3.4. *For each $\delta, C > 0$, there is $R > 0$ such that for any $n \in \mathbb{N}$, there exists a finite set $S \subset \mathbb{C}$ such that $|z| \in \mathbb{N}$ and $nR + c \leqslant |z| \leqslant (n + 1)R - c$ for any $z \in S$, $|z - z'| \geqslant c$ for any $z, z' \in S$, $z \neq z'$ and for each $w \in \mathbb{T}$, there exists $z \in S$ such that $|w - \frac{z}{|z|}| < \delta/|z|$.*

Proof. Without loss of generality, we may assume that $0 < \delta < 1$. Pick $m \in \mathbb{N}$ such that $2m \geqslant c$ and $h \in \mathbb{N}$ such that $h \geqslant (40 \cdot m)/\delta$. We shall show that $R = hm$ satisfies the desired condition. Pick $n \in \mathbb{N}$ and consider $k = k(n) \in \mathbb{N}$ defined by the formula $k = [\frac{\pi(n+1)m}{2\delta n}] + 1$, where $[t]$ is the integer part of $t \in \mathbb{R}$. For $1 \leqslant j \leqslant k$ let $n_j = nR + 2jm$. Clearly n_j are natural numbers and $n_1 = nR + 2m \geqslant nR + c$. On the other hand, $n_k = nR + 2mk \leqslant (n + 1)R - 2m$. Indeed, the last inequality is equivalent to $2(k + 1) \leqslant h$, which is an easy consequence of the two inequalities $h > (40 \cdot m)/\delta$ and $k + 1 \leqslant \frac{\pi(n+1)m}{2\delta n} + 2 \leqslant \frac{\pi m}{\delta} + 2$. Thus,

$$nR + c \leqslant n_1 \leqslant n_j \leqslant n_k \leqslant (n + 1)R - 2m \leqslant (n + 1)R - c \quad \text{for } 1 \leqslant j \leqslant k. \tag{3.1}$$

Now we can define a finite set S of complex numbers in the following way:

$$S = \{z_{j,l}\colon\ 1 \leqslant j \leqslant k,\ 0 \leqslant l \leqslant 2nh - 1\}, \quad \text{where } z_{j,l} = n_j \exp\left(\frac{\pi i(lk + j)}{nhk}\right) \tag{3.2}$$

and $\exp(z)$ stands for e^z. Clearly for each $z_{j,l} \in S$, we have $|z_{j,l}| = n_j \in \mathbb{N}$. Moreover, according to (3.1), $nR + c \leqslant |z| \leqslant (n + 1)R - c$ for any $z \in S$. Next, let $z, z' \in S$ and $z \neq z'$. Then $z = z_{j,l}$ and $z' = z_{p,q}$ for $1 \leqslant j, p \leqslant k$, $0 \leqslant l, q \leqslant 2nh - 1$ and $(j, l) \neq (p, q)$. If $j \neq p$, then $|z - z'| \geqslant ||z| - |z'|| = |n_j - n_p| = 2m|j - p| \geqslant 2m \geqslant c$. If $j = p$, then $l \neq q$ and

$$|z - z'| = n_j\left|\exp\left(\frac{\pi il}{nh}\right) - \exp\left(\frac{\pi iq}{nh}\right)\right| \geqslant n_j\left|\exp\left(\frac{\pi i}{nh}\right) - 1\right| = 2n_j \sin\left(\frac{\pi}{2nh}\right).$$

The inequality $\sin x \geqslant \frac{2x}{\pi}$ for $0 \leqslant x \leqslant \pi/2$, the inequality $n_j > nR$ and the equality $R = hm$ imply $|z - z'| \geqslant \frac{4\pi n_j}{2\pi nh} = \frac{2n_j}{nh} > \frac{2nR}{nh} = 2m \geqslant c$. Thus $|z - z'| \geqslant c$ for any $z, z' \in S$, $z \neq z'$. Finally, consider the set $\Sigma = \{z/|z|\colon z \in S\}$. Clearly

$$\Sigma = \left\{ \exp\left(\frac{\pi i(lk + j)}{nhk}\right)\colon \begin{matrix} 1 \leqslant j \leqslant k, \\ 0 \leqslant l \leqslant 2nh - 1 \end{matrix} \right\} = \left\{ \exp\left(\frac{\pi ij}{nhk}\right)\colon 1 \leqslant j \leqslant 2nhk \right\}$$
$$= \left\{ z \in \mathbb{C}\colon z^{2nhk} = 1 \right\}.$$

It immediately follows that

$$\sup_{w \in \mathbb{T}} \min_{z \in \Sigma} |w - z| = \left| 1 - \exp\left(\frac{\pi i}{2nhk}\right) \right| = 2\sin\left(\frac{\pi}{4nhk}\right) \leqslant \frac{\pi}{2nhk} = \frac{\pi m}{2nRk}.$$

Since $k > \frac{\pi(n+1)m}{2\delta n}$, we get $\sup_{w \in \mathbb{T}} \min_{z \in \Sigma} |w - z| < \delta(n+1)^{-1}R^{-1}$. That is, for any $w \in \mathbb{T}$, there exists $z \in S$ such that $|w - \frac{z}{|z|}| < \frac{\delta}{R(n+1)}$. Since $|z| < R(n+1)$, we obtain $|w - \frac{z}{|z|}| < \delta/|z|$, which completes the proof. □

Lemma 3.5. *Let X be a locally convex topological vector space and $\{T_z\}_{z \in \mathbb{C}}$ be an operator group on X such that the map $(u, h) \mapsto T_h u$ from $X \times \mathbb{C}$ to X is continuous. Let also $x \in X$ and p be a continuous seminorm on X. Then there exist a continuous seminorm q on X and $\delta > 0$ such that $p \leqslant q$ and for any $a \in \mathbb{R}$, $w \in \mathbb{T}$, $n \in \mathbb{N}$ and $y \in X$ satisfying $q(x - e^{an}T_{wn}y) < 1$, we have $p(x - e^{bn}T_{zn}y) < 1$ whenever $b \in \mathbb{R}$ and $z \in \mathbb{T}$ are such that $|a - b| < \delta/n$ and $|w - z| < \delta/n$.*

Proof. Since the map $(u, h) \mapsto T_h u$ from $X \times \mathbb{C}$ to X is continuous, there is $\theta > 0$ and a continuous seminorm q on X such that $p(x - T_h x) \leqslant 1/4$ and $p(T_h u) \leqslant q(u)/4$ for any $u \in X$ whenever $|h| \leqslant \theta$. In particular, $p(u) \leqslant q(u)/4 \leqslant q(u)$ for each $u \in X$. Pick $r \in (0, \theta)$ and assume that $a, b \in \mathbb{R}$, $w, z \in \mathbb{T}$, $n \in \mathbb{N}$ and $y \in X$ are such that $q(x - e^{an}T_{wn}y) < 1$, $|a - b| < r/n$ and $|w - z| < r/n$. Then $p(e^{an}T_{wn}y) \leqslant q(e^{an}T_{wn}y) \leqslant q(x) + 1$. Since $|a - b| < r/n$, we have $|e^{(b-a)n} - 1| < e^r - 1$. Hence

$$p\big(e^{bn}T_{wn}y - e^{an}T_{wn}y\big) = \big|e^{(b-a)n} - 1\big| p\big(e^{an}T_{wn}y\big) \leqslant \big(e^r - 1\big)\big(q(x) + 1\big). \tag{3.3}$$

Since $|nw - nz| < r < \theta$ and $p(T_h u) \leqslant q(u)/4$ for any $u \in X$ whenever $|h| \leqslant \theta$, we have

$$p\big(T_{(z-w)n}x - e^{an}T_{zn}y\big) = p\big(T_{(z-w)n}\big(x - e^{an}T_{wn}y\big)\big) \leqslant q\big(x - e^{an}T_{wn}y\big)/4 < 1/4.$$

Since $|(z - w)n| < r < \theta$, we get $p(x - T_{(z-w)n}x) \leqslant 1/4$. Using this inequality together with the last display and the triangle inequality, we obtain $p(x - e^{an}T_{zn}y) \leqslant 1/2$. The latter together with (3.3) and the triangle inequality gives $p(x - e^{bn}T_{zn}y) < (e^r - 1)(q(x) + 1) + 1/2$. Hence any $\delta \in (0, \theta)$ satisfying $(e^\delta - 1)(q(x) + 1) < 1/2$, satisfies also the desired condition. □

3.1. Proof of Theorem 3.3

By Theorems LM and CMP, $\mathfrak{H}(bT_a) = \mathfrak{H}(b'T_{a'})$ if $|b| = |b'|$ and $a/a' \in \mathbb{R}_+$. Hence the set of common hypercyclic vectors of the family $\{aT_b\colon a \in \mathbb{K}^\star,\ b \in \mathbb{C}^\star\}$ coincides with the set G of common hypercyclic vectors for the family $\{e^b T_a\colon (a, b) \in \mathbb{T} \times \mathbb{R}\}$. Thus it remains to show that

G is a dense G_δ-subset of X. Fix $d > 0$. According to Corollary 2.3, it suffices to demonstrate that

$$\begin{aligned}&\text{for any non-empty open subsets } U \text{ and } V \text{ of } X,\ \text{there is } y \in U \text{ such that}\\ &\text{for any } a \in \mathbb{T} \text{ and } b \in [-d, d] \text{ there is } n \in \mathbb{N} \text{ for which } e^{bn} T_{an} y \in V.\end{aligned} \tag{3.4}$$

Pick a continuous seminorm p on X and $u, x \in X$ such that $\{y \in X\colon\ p(u - y) < 1\} \subseteq U$ and $\{y \in X\colon\ p(x - y) < 1\} \subseteq V$. By the uniform boundedness principle [21], strong continuity of $\{T_z\}_{z\in\mathbb{C}}$ implies that the map $(z, v) \mapsto T_z v$ from $\mathbb{C} \times X$ to X is continuous. By Lemma 3.5, there is a continuous seminorm q on X and $\delta > 0$ such that $p(v) \leqslant q(v)$ for any $v \in X$ and

$$\begin{aligned}&\text{for any } a, b \in \mathbb{R},\ w, z \in \mathbb{T},\ n \in \mathbb{N} \text{ and } y \in X \text{ satisfying } q\big(x - e^{an} T_{wn} y\big) < 1,\\ &|a - b| < \delta/n \text{ and } |w - z| < \delta/n,\ \text{we have } p\big(g - e^{bn} T_{zn} y\big) < 1.\end{aligned} \tag{3.5}$$

Since $\{T_z\}_{z\in\mathbb{C}}$ has the Runge property, there is $c > 0$ such that

$$\begin{aligned}&\text{for any finite set } S \subset \mathbb{C} \text{ with } |z - z'| \geqslant c \text{ for } z, z' \in S,\ z \neq z',\ \text{any } \varepsilon > 0 \text{ and any}\\ &\{x_z\}_{z\in S} \in X^S,\ \text{there exists } y \in X \text{ such that } q(T_z y - x_z) < \varepsilon \text{ for any } z \in S.\end{aligned} \tag{3.6}$$

Let $R > 0$ be the number provided by Lemma 3.4 for the just chosen δ and c. By Lemma 3.4, for each $n \in \mathbb{N}$ there is a finite set $S_n \subset \mathbb{C}$ such that $|z| \in \mathbb{N}$ and $nR + c \leqslant |z| \leqslant (n + 1)R - c$ for any $z \in S_n$, $|z - z'| \geqslant c$ for any $z, z' \in S_n$, $z \neq z'$ and for each $w \in \mathbb{T}$, there is $z \in S_n$ such that $|w - \frac{z}{|z|}| < \frac{\delta}{|z|}$. Since $\sum_{n=1}^{\infty} n^{-1} = \infty$, we can pick $d_1, \ldots, d_k \in [-d, d]$ for which

$$[-d, d] \subseteq \bigcup_{n=1}^{k} \left(d_n - \frac{\delta R^{-1}}{n+1}, d_n + \frac{\delta R^{-1}}{n+1}\right). \tag{3.7}$$

Let $S = \bigcup_{n=1}^{k} S_n$ and $\Lambda = S \cup \{0\}$. It is straightforward to see that Λ is a finite set, $|z| \in \mathbb{Z}_+$ for any $z \in \Lambda$ and $|z - u| \geqslant c$ for any $z, u \in \Lambda$, $z \neq u$. Let $N = \max\{|z|\colon\ z \in \Lambda\}$ and $\varepsilon = d^{-N}$. By (3.6), there is $y \in X$ such that $q(u - y) < \varepsilon$ and $q(T_z y - e^{-c_n|z|} x) < \varepsilon$ for each $z \in S$. Then $p(u - y) \leqslant q(u - y) < \varepsilon < 1$ and therefore $f \in U$. By definition of ε, $q(x - e^{c_n|z|} T_z y) < 1$ for each $z \in S$. Let now $a \in \mathbb{T}$ and $b \in [-d, d]$. According to (3.7), there is $n \in \{1, \ldots, k\}$ such that $|b - d_n| < \frac{\delta R^{-1}}{n+1}$. By the mentioned property of the set S_n, we can choose $z \in S_n$ such that $|a - \frac{z}{|z|}| < \frac{\delta}{|z|}$. Since $|z| < R(n + 1)$, we have $|b - d_n| < \frac{\delta}{|z|}$. By (3.5), $p(x - e^{b|z|} T_{a|z|} y) < 1$. Hence $e^{b|z|} T_{a|z|} f \in V$, which completes the proof of (3.4) and that of Theorem 3.3.

4. Scalar multiples of a fixed operator

In this section we shall prove Theorems 1.6 and 1.7 as well as Corollaries 1.8, 1.9 and 1.10. Recall that a subset A of a vector space is called *balanced* if $zx \in A$ for any $x \in A$ and $z \in \mathbb{K}$ satisfying $|z| \leqslant 1$. It is well known that any topological vector space has a base of open neighborhoods of zero consisting of balanced sets. For two subsets A, B of a vector space X we say that *A absorbs B* if there exists $c > 0$ such that $B \subseteq zA$ for any $z \in \mathbb{K}$ satisfying $|z| \geqslant c$. Obviously, if A is balanced, then A absorbs B if and only if there is $c > 0$ for which $B \subseteq cA$.

Lemma 4.1. *Let X be a topological vector space and U be a non-empty open subset of X. Then there exist a non-empty open subset V of X and a balanced neighborhood W of zero in X such that $V + W \subseteq U$ and W absorbs V.*

Proof. Pick $u \in U$ and a balanced neighborhood W_0 of zero in X such that $u + W_0 + W_0 + W_0 \subseteq U$. Denote $V = u + W_0$ and $W = W_0 + W_0$. Clearly V is a non-empty open subset of X, W is a balanced neighborhood of 0 in X and $V + W = u + W_0 + W_0 + W_0 \subseteq U$. Since W_0 is a neighborhood of 0 in X, we can pick $c \geqslant 1$ such that $u \in cW_0$. Since W_0 is balanced and $c \geqslant 1$, $W_0 \subseteq cW_0$ and therefore $V = u + W_0 \subseteq cW_0 + W_0 \subseteq c(W_0 + W_0) = cW$. Since W is balanced, W absorbs V. $\square$

To any continuous linear operator T on a complex topological vector space X there corresponds $\mathbf{T} \in \mathcal{L}_{\mathbb{R},\mathbb{T}}(X, X)$ defined by the formula $\mathbf{T}_{t,w,n}x = we^{tn}T^n x$. We will use the symbol $M(T, u, \Lambda, U)$ to denote the sets defined in (2.3) for $\mathbf{T}$. In other words, for $\Lambda \subseteq \mathbb{Z}_+$, $t \in \mathbb{R}$, $u \in X$ and a subset U of X, we write

$$M(T, u, \Lambda, U) = \big\{t \in \mathbb{R}\colon\ we^{tn}T^n u \in U \text{ for some } n \in \Lambda \text{ and } w \in \mathbb{T}\big\}.$$

Lemma 4.2. *Let X be a complex topological vector space, W be a balanced neighborhood of* 0 *in X, $c > 0$, $k \in \mathbb{N}$ and $\delta \in (0, (2ck)^{-1}]$. Then for any $m \in \mathbb{N}$, any $\alpha \in [-c, c]$, any $w \in \mathbb{T}$, any neighborhood W_0 of zero in X and any $x \in cW$ such that $T^k x = we^{-\alpha k}x$, there exist $u \in W_0$ and a finite set $\Lambda \subset \mathbb{N}$ such that* $\min \Lambda \geqslant m$ *and* $[\alpha + \delta, \alpha + 2\delta] \subseteq M(T, u, \Lambda, x + W)$.

Proof. Let $\alpha \in [-c, c]$, $w \in \mathbb{T}$ and any $x \in cW$ be such that $T^k x = we^{-\alpha k}x$. For each $p \in \mathbb{N}$ consider $u_p = e^{-2\delta kp}x$. Since $T^k x = we^{-\alpha k}x$, we see that for $0 \leqslant j \leqslant p$,

$$T^{(p+j)k}u_p = e^{-\alpha(p+j)k}e^{-2\delta kp}w^{p+j}x = \exp\left(-(p+j)k\left(\alpha + \frac{2p\delta}{p+j}\right)\right)w^{p+j}x.$$

That is,

$$w_j e^{(p+j)k\theta_j}T^{(p+j)k}u_p = x \quad \text{for } 1 \leqslant j \leqslant p, \text{ where } \theta_j = \alpha + \frac{2\delta p}{p+j} \text{ and } w_j = w^{-p-j} \in \mathbb{T}. \tag{4.1}$$

Let now $0 \leqslant l \leqslant p - 1$ and $\theta \in [\theta_{l+1}, \theta_l]$. Since

$$e^{(p+l)k\theta}T^{(p+l)k}u_p = e^{(p+l)k(\theta-\theta_l)}e^{(p+l)k\theta_l}T^{(p+l)k}u_p,$$

using (4.1) with $j = l$, we obtain

$$w_l e^{(p+l)k\theta}T^{(p+l)k}u_p = e^{(p+l)k(\theta-\theta_l)}x = x + \big(e^{(p+l)k(\theta-\theta_l)} - 1\big)x.$$

Taking into account that $-(\theta_l - \theta_{l+1}) \leqslant \theta - \theta_l \leqslant 0$ and using the inequality $0 \leqslant 1 - e^{-t} \leqslant t$ for $t \geqslant 0$, we see that $|e^{(p+l)k(\theta-\theta_l)} - 1| \leqslant (p + l)k(\theta_l - \theta_{l+1})$. This inequality, the inclusion $x \in cW$ the last display and the fact that W is balanced imply that

$$w_l e^{(p+l)k\theta}T^{(p+l)k}u_p \in x + c\big|e^{(p+l)k(\theta-\theta_l)} - 1\big|W \subseteq x + c(p + l)k(\theta_l - \theta_{l+1})W.$$

Since $\theta_l - \theta_{l+1} = \frac{2p\delta}{(p+l)(p+l+1)} \leqslant \frac{2\delta}{p+l}$ and $\delta \leqslant (2ck)^{-1}$, we have $c(p+l)k(\theta_l - \theta_{l+1}) \leqslant 1$. Thus according to the above display, $w_l e^{(p+l)k\theta} T^{(p+l)k} u_p \in x + W$ whenever $\theta \in [\theta_{l+1}, \theta_l]$. It follows that $[\theta_{l+1}, \theta_l] \subseteq M(T, u_p, \Lambda_p, x + W)$ for $0 \leqslant l \leqslant p-1$, where $\Lambda_p = \{(p+j)k\colon 0 \leqslant j \leqslant p\}$. Since the sequence $\{\theta_j\}_{0\leqslant j\leqslant p}$ decreases, $\theta_0 = \alpha + 2\delta$ and $\theta_p = \alpha + \delta$, we see that $[\alpha+\delta, \alpha+2\delta] = \bigcup_{l=0}^{p-1} [\theta_{l+1}, \theta_l]$. Since $[\theta_{l+1}, \theta_l] \subseteq M(T, u_p, \Lambda_p, x + W)$ for $0 \leqslant l \leqslant p-1$, we have $[\alpha+\delta, \alpha+2\delta] \subseteq M(T, u_p, \Lambda_p, x+W)$ for any $p \in \mathbb{N}$. Clearly $\min \Lambda_p = pk \to \infty$ and $u_p = e^{-2\delta k p} x \to 0$ in X as $p \to \infty$. Thus we can pick $p \in \mathbb{N}$ such that $\min \Lambda_p > m$ and $u_p \in W_0$. Then $u = u_p$ and $\Lambda = \Lambda_p$ for such a p satisfy all desired conditions. $\square$

We shall prove a statement more general than Theorem 1.6.

Theorem 4.3. *Let X be a separable complex $\mathcal{F}$-space, $T \in L(X)$ and $0 \leqslant a < b \leqslant \infty$. Assume also that the following condition is satisfied.*

(4.3.1) *For any compact interval $J \subset (a,b)$ and any non-empty open subset V of X, there exist $k = k(J,V) \in \mathbb{N}$ and a dense subset $C = C(J,V)$ of J such that*

$$V \cap \bigcup_{w\in\mathbb{T}} \mathtt{ker}\bigl(T^k - wc^k I\bigr) \neq \varnothing \quad \textit{for each } c \in C.$$

Then $\mathfrak{H}\{zT\colon b^{-1} < |z| < a^{-1}\}$ is a dense G_δ-set.

Proof. Let $\alpha_0, \alpha, \beta \in \mathbb{R}$ be such that $b^{-1} < e^{\alpha_0} < e^{\alpha} < e^{\beta} < a^{-1}$. For each $\omega \in [\alpha, \beta]$ consider the family $\mathcal{F}_\omega = \{ze^{\omega n}T^n\colon z \in \mathbb{T},\ n \in \mathbb{Z}_+\}$. We shall apply Corollary 2.6 with $A = \mathbb{T}$, $T_{\omega,a,n} = ae^{\omega n}T^n$ and $\Omega = [\alpha, \beta]$. First, pick a compact interval $J \subset (a, e^{-\beta})$. For each non-empty open subset V_0 of X, we can use (4.3.1) to find $x \in V_0$, $k \in \mathbb{N}$, $r \in J$ and $w \in \mathbb{T}$ such that $T^k x = wr^k x$. The latter equality implies that x is a sum of finitely many eigenvectors of T corresponding to eigenvalues λ_j with $|\lambda_j| = r < e^{-\beta}$. Hence $e^{\beta n}T^n x \to 0$ as $n \to \infty$. Since V_0 is an arbitrary non-empty open subset of X and $x \in V_0$, we see that the space $E = \{x \in X\colon e^{\beta n}T^n x \to 0\}$ is dense in X. It immediately follows that

$$\text{for any } x \in E,\ ze^{\omega n}T^n x \to 0 \text{ as } n \to \infty \text{ uniformly for } (z,\omega) \in \mathbb{T} \times [\alpha, \beta].$$

Hence (2.5.1) is satisfied. Let now U be a non-empty open subset of X. By Lemma 4.1, there exist a balanced neighborhood W of zero in X and a non-empty open subset V of X such that $V + W \subseteq U$ and W absorbs V. Since W absorbs V, there is $c > 0$ such that $V \subseteq cW$. According to (4.3.2), we can pick $k \in \mathbb{N}$ and a dense subset R of $[\alpha_0, \beta]$ for which

$$V \cap \bigcup_{w\in\mathbb{T}} \mathtt{ker}\bigl(T^k - we^{-rk} I\bigr) \neq \varnothing \quad \text{for any } r \in R. \tag{4.2}$$

Let $\delta_0 = \min\{(2ck)^{-1}, \alpha - \alpha_0\}$ and $r \in R$. By (4.2), we can pick $w_r \in \mathbb{T}$ and $x_r \in V \subseteq cW$ such that $T^k x_r = wr^{-rk}x_r$. By Lemma 4.2, for any neighborhood W_0 of zero in X and any $m \in \mathbb{N}$, there exist $u \in W_0$ and a finite set $\Lambda \subset \mathbb{N}$ satisfying $\min \Lambda \geqslant m$ and $[r + \delta_0, r + 2\delta_0] \subseteq M(T, u, \Lambda, x_r + W)$. Pick $\delta \in (0, \delta_0)$. Since R is dense in $[\alpha_0, \beta]$ and $\delta_0 \leqslant \alpha - \alpha_0$, it is easy to see that each compact interval $J \subseteq [\alpha, \beta]$ of length at most δ is contained in $[r + \delta_0, r + 2\delta_0]$ for some $r \in R$. Thus for each compact interval $J \subseteq [\alpha, \beta]$ of length at most δ, any neighborhood W_0 of zero in X and any $m \in \mathbb{N}$, there exist $r \in R$, $u \in W_0$ and a finite set Λ such that $\min \Lambda \geqslant m$ and $J \subseteq M(T, u, \Lambda, x_r + W)$. The latter inclusion means that for each $t \in J$, there exist $w_t \in \mathbb{T}$ and $n_t \in \Lambda$ such that $w_t T^{n_t} u \in x_r + W$. Since $x_r \in V$ and $V + W \subseteq U$, we get $w_t T^{n_t} u \in U$. That is, for any compact interval $J \subseteq [\alpha, \beta]$ of length at most δ, any neighborhood W_0 of zero in X and any $m \in \mathbb{N}$, there exist $u \in W_0$ and a finite set Λ such that $\min \Lambda \geqslant m$ and $J \subseteq M(T, u, \Lambda, U)$. Thus (2.6.2) is also satisfied. By Corollary 2.6,

$$H_{\alpha,\beta} = \bigcap_{\omega \in [\alpha,\beta]} \mathfrak{U}(\mathcal{F}_\omega) \text{ is a dense } G_\delta\text{-subset of } X \text{ whenever } b^{-1} < e^\alpha < e^\beta < a^{-1}.$$

By Theorem LM, $\mathfrak{U}(\mathcal{F}_\omega) = \mathfrak{H}(ze^\omega T)$ for any $\omega \in \mathbb{R}$ and $z \in \mathbb{T}$. Hence $H_{\alpha,\beta} = \mathfrak{H}\{zT\colon e^\alpha \leqslant |z| \leqslant e^\beta\}$. From the above display it now follows that $\mathfrak{H}\{zT\colon b^{-1} < |z| < a^{-1}\}$ is a dense G_δ-subset of X as the intersection of a countable family of dense G_δ-sets. □

4.1. Proof of Theorem 1.6

We shall prove Theorem 1.6 by means of applying Theorem 4.3. To do this it suffices to demonstrate that (4.3.1) is satisfied. Let $J \subset (a, b)$ be a compact interval and V be a non-empty open subset of X. For any $k \in \mathbb{N}$ let $O_k = \{c \in (a, b)\colon F_{k,c} \cap V \neq \emptyset\}$. By (1.6.2), O_k are open subsets of (a, b). According to (1.6.3), $\{O_k\colon k \in \mathbb{N}\}$ is an open covering of (a, b). Since J is compact, we can pick $k_1, \dots, k_n \in \mathbb{N}$ such that $J \subseteq \bigcup_{j=1}^n O_{k_j}$. By (1.6.4), there is $k \in \mathbb{N}$ for which $\bigcup_{j=1}^n F_{k_j,c} \subseteq F_{k,c}$ for any $c \in (a, b)$. Hence $O_k \supseteq \bigcup_{j=1}^n O_{k_j} \supseteq J$. It follows that for any $c \in J$, there is $x \in F_{k,c} \cap V$. According to (1.6.1), there is $w \in \mathbb{T}$ for which $x \in \mathtt{ker}(T^k - wc^k I)$. Thus $V \cap \bigcup_{w \in \mathbb{T}} \mathtt{ker}(T^k - wc^k I) \neq \emptyset$ for any $c \in J$. That is, (4.3.1) is satisfied with $C = J$. It remains to apply Theorem 4.3 to conclude the proof of Theorem 1.6.

4.2. Proof of Theorem 1.7

Recall that a map h from a topological space X to a topological space Y is called *open* if $h(U)$ is open in Y for any open subset U of X. Recall also that a subset A of a connected open subset U of $\mathbb{C}^m$ is called a *set of uniqueness* if any holomorphic function $\varphi : U \to \mathbb{C}$ vanishing on A is identically zero. The following lemma contains few classical results that can be found in virtually any book on complex analysis.

Lemma 4.4. *Let $m \in \mathbb{N}$ and U be a connected open subset of $\mathbb{C}^m$. Then any non-empty open subset of U is a set of uniqueness and any non-constant holomorphic map $\varphi : U \to \mathbb{C}$ is open. Moreover, if $m = 1$, then any subset of U with at least one limit point in U is a set of uniqueness.*

We need the following generalization of the last statement of Lemma 4.4 to the case $m > 1$. Although it is probably known, the author was unable to locate a reference.

Lemma 4.5. *Let $m \in \mathbb{N}$, U be a connected open subset of $\mathbb{C}^m$, $\varphi : U \to \mathbb{C}$ be a non-constant holomorphic map and A be a subset of $\mathbb{C}$ with at least one limit point in $\varphi(U)$. Then $\varphi^{-1}(A)$ is a set of uniqueness. In particular, if $a = \inf_{z \in U} |\varphi(z)|$, $b = \sup_{z \in U} |\varphi(z)|$, $c \in (a, b)$ and G is a dense subset of $\mathbb{T}$, then $\varphi^{-1}(cG)$ is a set of uniqueness.*

Proof. Assume the contrary. Then there exists a non-zero holomorphic function $f : U \to \mathbb{C}$ such that $f|_{\varphi^{-1}(A)} = 0$. Let $a \in \varphi(U)$ be a limit point of A and $w \in U$ be such that $\varphi(w) = a$. Pick a convex open subset V of $\mathbb{C}^m$ such that $w \in V \subseteq U$. For any complex one-dimensional linear subspace L of $\mathbb{C}^m$, $V_L = (w + L) \cap V$ can be treated as a convex open subset of $\mathbb{C}$. If $\varphi_L = \varphi|_{V_L}$ is non-constant, then by Lemma 4.4, $\varphi_L : V_L \to \mathbb{C}$ is open. Since $a = \varphi(w)$ is a limit point of A, it follows that w is a limit point of $\varphi_L^{-1}(A)$. Using the one-dimensional uniqueness theorem, we see that $\varphi_L^{-1}(A)$ is a set of uniqueness in V_L. Since f vanishes on $\varphi^{-1}(A) \supseteq \varphi_L^{-1}(A)$, $f|_{V_L} = 0$. On the other hand, if φ_L is constant, then $(\varphi - a)|_{V_L} = 0$. Since L is arbitrary, we have $f(\varphi - a)|_V = 0$. Since V, being a non-empty open subset of U, is a set of uniqueness, we have $f \cdot (\varphi - a) = 0$. Since $f \not\equiv 0$, there is a non-empty open subset W of U such that $f(z) \neq 0$ for any $z \in W$. The equality $f \cdot (\varphi - a) = 0$ implies that $\varphi(z) = a$ for any $z \in W$. Since W is a set of uniqueness, $\varphi \equiv a$. We have arrived to a contradiction. Thus $\varphi^{-1}(A)$ is a set of uniqueness.

Assume now that $a = \inf_{z \in U} |\varphi(z)|$, $b = \sup_{z \in U} |\varphi(z)|$, $c \in (a, b)$ and G is a dense subset of $\mathbb{T}$. Since U is connected $c\mathbb{T} \cap \varphi(U) \neq \emptyset$. Since φ is open, the set $\varphi(U)$ is open in $\mathbb{C}$. Thus density of G in $\mathbb{T}$ implies that $cG \cap \varphi(U)$ is dense in $c\mathbb{T} \cap \varphi(U)$, which is an open subset of $c\mathbb{T}$. Hence cG has plenty of limit points in $\varphi(U)$ and it remains to apply the first part of the lemma. $\square$

We shall prove Theorem 1.7 by means of applying Theorem 1.6. First, note that density of $\operatorname{span}\{f(z)\colon z \in U\}$ implies separability of X. Let

$$F_{k,c} = \operatorname{span}\bigl\{f(z)\colon z \in U,\ \varphi(z)^k = c^k\bigr\} \quad \text{for } k \in \mathbb{N} \text{ and } c \in (a, b).$$

In order to apply Theorem 1.6 it suffices to verify that the map $(k, c) \mapsto F_{k,c}$ satisfies conditions (1.6.1)–(1.6.4). First, from the equality $Tf(z) = \varphi(z) f(z)$ it follows that $T^k x = c^k x$ for any $x \in F_{k,c}$. Hence (1.6.1) is satisfied. Clearly $F_{k,c} \subseteq F_{m,c}$ whenever k is a divisor of m. Hence for any $c \in (a, b)$ and any $k_1, \dots, k_n \in \mathbb{N}$, $F_{k_j,c} \subseteq F_{k,c}$ for $1 \leqslant j \leqslant n$, where $k = k_1 \cdot \dots \cdot k_n$. Thus (1.6.4) is satisfied. It is easy to see that

$$F_c = \bigcup_{k=1}^{\infty} F_{k,c} = \operatorname{span}\bigl\{f(z)\colon \varphi(z) \in c\mathbb{G}\bigr\}, \quad \text{where } \mathbb{G} = \bigl\{z \in \mathbb{T}\colon z^k = 1 \text{ for some } k \in \mathbb{N}\bigr\}.$$

In order to prove (1.6.3), we have to show that F_c is dense in X. Assume the contrary. Since F_c is a vector space and X is locally convex, we can pick $g \in X^*$ such that $g \neq 0$ and $g(x) = 0$ for each $x \in F_c$. In particular, $g(f(z)) = 0$ whenever $\varphi(z) \in c\mathbb{G}$. By Lemma 4.5, $\varphi^{-1}(c\mathbb{G})$ is a set of uniqueness. Since the holomorphic function $g \circ f$ vanishes on $\varphi^{-1}(c\mathbb{G})$, it is identically zero. Hence $g(f(z)) = 0$ for any $z \in U$, which contradicts the density of $\operatorname{span}\{f(z)\colon z \in U\}$ in X. This contradiction completes the proof of (1.6.3). It remains to verify (1.6.2). Let $k \in \mathbb{N}$, V be a non-empty open subset of X and $G = \{c \in (a, b)\colon F_{k,c} \cap V \neq \emptyset\}$. We have to show that G is open in $\mathbb{R}$. Let $c \in G$. Then there exist $z_1, \dots, z_n \in U$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that $\varphi(z_j)^k = c^k$ for $1 \leqslant j \leqslant n$ and $\sum_{j=1}^n \lambda_j f(z_j) \in V$. Since f is continuous, we can pick $\varepsilon > 0$ such that $z_j + \varepsilon\mathbb{D}^m \subset U$ for $1 \leqslant j \leqslant n$ and $\sum_{j=1}^n \lambda_j f(w_j) \in V$ for any choice of $w_j \in z_j + \varepsilon\mathbb{D}^m$. By Lemma 4.4, φ is open and therefore there exists $\delta > 0$ such that $\varphi(z_j) + c\delta\mathbb{D} \subseteq \varphi(z_j + \varepsilon\mathbb{D}^m)$ for $1 \leqslant j \leqslant n$. In particular, since $|\varphi(z_j)| = c$, we see that $(1 - \delta, 1 + \delta)\varphi(z_j) \subset \varphi(z_j + \varepsilon\mathbb{D})$ for $1 \leqslant j \leqslant n$. Hence for each $s \in (1 - \delta, 1 + \delta)$, we can pick $w_1, \dots, w_n \in U$ such that $w_j \in z_j + \varepsilon\mathbb{D}^m$ and $\varphi(w_j) = s\varphi(z_j)$ for $1 \leqslant j \leqslant n$. Then $\varphi(w_j)^k = s^k \varphi(z_j)^k = (cs)^k$ and $\sum_{j=1}^n \lambda_j f(w_j) \in V$ since $w_j \in z_j + \varepsilon\mathbb{D}$. Hence $cs \in G$ for each $s \in (1 - \delta, 1 + \delta)$ and therefore c is an interior point of G. Since c is an arbitrary point of G, G is open. This completes the proof of (1.6.2). It remains to apply Theorem 1.6 to conclude the proof of Theorem 1.7.

4.3. *Proof of Corollary 1.8*

Note that $\mathcal{H}^*$ with the usual norm is a Banach space. Consider the map $f : U \to \mathcal{H}^*$ defined by the formula $f(z)(x) = x(z)$. It is straightforward to verify that f is holomorphic, $M_\varphi^* f(z) = \varphi(z) f(z)$ for each $z \in U$ and $\mathtt{span}\{f(z)\colon z \in U\}$ is dense in $\mathcal{H}^*$. The latter is a consequence of the fact that evaluation functionals separate points of $\mathcal{H}$. Using Theorem 1.7, we immediately obtain that $G_0 = \mathfrak{H}\{zM_\varphi^*\colon b^{-1} < |z| < a^{-1}\}$ is a dense G_δ-subset of $\mathcal{H}^*$. Now consider the map $R : \mathcal{H} \to \mathcal{H}^*$, $Rx(y) = \langle y, x\rangle$, where $\langle\cdot,\cdot\rangle$ is the scalar product of the Hilbert space $\mathcal{H}$. According to the Riesz theorem, R is an $\mathbb{R}$-linear isometric isomorphism (it happens to be complex conjugate linear). It is also easy to see that $R^{-1}S^*R = S^\star$ for any $S \in L(\mathcal{H})$, where S^* is the dual of S and $S^\star$ is the Hilbert space adjoint of S. Hence $G = R^{-1}(G_0)$, where $G = \mathfrak{H}\{zM_\varphi^\star\colon b^{-1} < |z| < a^{-1}\}$. Since R is a homeomorphism from $\mathcal{H}$ onto $\mathcal{H}^*$, G is a dense G_δ-subset of $\mathcal{H}$.

4.4. *Proof of Corollary 1.9*

Consider the map $f : \mathbb{C} \to \mathcal{H}(\mathbb{C})$ defined by the formula $f(w)(z) = e^{wz}$. It is easy to see that f is holomorphic, $\mathtt{span}\{f(z)\colon z \in \mathbb{C}\}$ is dense in $\mathcal{H}(\mathbb{C})$ and for each $w \in \mathbb{C}$, $\mathtt{ker}(D - wI) = \mathtt{span}\{f(w)\}$. In particular, $Df(w) = wf(w)$ and using the equality $TD = DT$, we get $wTf(w) = DTf(w)$ for each $w \in \mathbb{C}$. Hence $Tf(w) \in \mathtt{ker}(D - wI) = \mathtt{span}\{f(w)\}$ for any $w \in \mathbb{C}$. Thus there exists a unique function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $Tf(w) = \varphi(w)f(w)$ for each $w \in \mathbb{C}$. Using the fact that f is holomorphic and each $f(w)$ does not take value 0, one can easily verify that φ is holomorphic. Moreover, since T is not a scalar multiple of identity, φ is non-constant. By the Picard theorem, any non-constant entire function takes all complex values except for maybe one. Hence $\inf_{w\in\mathbb{C}} |\varphi(w)| = 0$ and $\sup_{w\in\mathbb{C}} |\varphi(w)| = \infty$. By Theorem 1.7, $\mathfrak{H}\{zT\colon z \in \mathbb{C}^\star\}$ is a dense G_δ-subset of $\mathcal{H}(\mathbb{C})$.

4.5. *Proof of Corollary 1.10*

First, we consider the case $\mathbb{K} = \mathbb{C}$. Let $a < \alpha < \beta < b$. By the assumptions, there is a dense subset E of X and a map $S : E \to E$ such that $TSx = x$, $\alpha^{-n}T^n x \to 0$ and $\beta^n S^n x \to 0$ for each $x \in E$. Let $U = \{w \in \mathbb{C}\colon \alpha < |w| < \beta\}$. Since X is locally convex and complete, the relations $\alpha^{-n}T^n x \to 0$ and $\beta^n S^n x \to 0$ ensure that for each $w \in U$, the series $\sum_{n=1}^\infty w^{-n}T^n x$ and $\sum_{n=1}^\infty w^n S^n x$ converge in X for any $x \in E$. Thus we can define

$$u_{x,w} = x + \sum_{n=1}^{\infty} \bigl(w^{-n}T^n x + w^n S^n x\bigr) \quad \text{for } w \in U \text{ and } x \in E.$$

Using the relations $TSx = x$ for $x \in E$ and $T \in L(X)$, one can easily verify that $Tu_{x,w} = wu_{x,w}$ for each $x \in E$ and $w \in U$. Now we consider

$$F_{k,c} = \mathtt{span}\bigl\{u_{x,w}\colon x \in E,\ w^k = c^k\bigr\} \quad \text{for } k \in \mathbb{N} \text{ and } c \in (\alpha, \beta).$$

We shall show that $F_{k,c}$ for $k \in \mathbb{N}$ and $c \in (\alpha, \beta)$ satisfy conditions (1.6.1)–(1.6.4). First, the equality $Tu_{x,w} = wu_{x,w}$ implies that $T^k y = c^k y$ for any $y \in F_{k,c}$. Hence (1.6.1) is satisfied. Clearly $F_{k,c} \subseteq F_{m,c}$ whenever k is a divisor of m. Hence for any $c \in (\alpha, \beta)$ and any $k_1, \dots, k_n \in \mathbb{N}$, $F_{k_j,c} \subseteq F_{k,c}$ for $1 \leqslant j \leqslant n$, where $k = k_1 \cdot \dots \cdot k_n$. Thus (1.6.4) is satisfied. It is easy to see that

$$F_c = \bigcup_{k=1}^{\infty} F_{k,c} = \mathtt{span}\{u_{x,w}\colon\ x \in E,\ w \in c\mathbb{G}\}, \quad \text{where } \mathbb{G} = \bigl\{z \in \mathbb{T}\colon\ z^k = 1 \text{ for some } k \in \mathbb{N}\bigr\}.$$

In order to prove (1.6.3), we have to show that F_c is dense in X. Assume the contrary. Since F_c is a vector space and X is locally convex, we can pick $g \in X^*$ such that $g \neq 0$ and $g(y) = 0$ for each $y \in F_c$. Hence for any $x \in E$ and $w \in c\mathbb{G}$, we have $f_x(w) = 0$, where $f_x(w) = g(u_{x,w})$. It is easy to verify that for any $x \in E$, the function $f_x : U \to \mathbb{C}$ is holomorphic. Since f_x vanishes on $c\mathbb{G}$, the uniqueness theorem implies that each f_x is identically zero. On the other hand, the 0th Laurent coefficient of f_x is $g(x)$. Hence $g(x) = 0$ for any $x \in E$. Since E is dense in X, we get $g = 0$. This contradiction completes the proof of (1.6.3). It remains to verify (1.6.2). Let $k \in \mathbb{N}$, V be a non-empty open subset of X and $G = \{c \in (\alpha, \beta)\colon\ F_{k,c} \cap V \neq \varnothing\}$. We have to show that G is open in $\mathbb{R}$. Let $c \in G$. Then there exist $x_1, \dots, x_n \in E$ and $w_1, \dots, w_n, \lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that $w_j^k = c^k$ for $1 \leqslant j \leqslant n$ and $\sum_{j=1}^n \lambda_j u_{x_j,w_j} \in V$. Since for any fixed $x \in E$, the map $w \mapsto u_{x,w}$ is continuous, there is $\delta > 0$ such that $y_s \in V$ if $|c - s| < \delta$, where $y_s = \sum_{j=1}^n \lambda_j u_{x_j, sw_j/c}$. On the other hand, $y_s \in E_{k,s}$ for each s and therefore $(c - \delta, c + \delta) \cap (\alpha, \beta) \subseteq G$. Hence c is an interior point of G. Since c is an arbitrary point of G, G is open. This completes the proof of (1.6.2). By Theorem 1.7, $\mathfrak{H}\{zT\colon\ \beta^{-1} < |z| < \alpha^{-1}\}$ is a dense G_δ-set whenever $a < \alpha < \beta < b$. Hence the set of common hypercyclic vectors of the family $\{zT\colon\ b^{-1} < |z| < a^{-1}\}$ is a dense G_δ-subset of X as a countable intersection of dense G_δ-sets. The proof of Corollary 1.10 in the case $\mathbb{K} = \mathbb{C}$ is complete.

Assume now that $\mathbb{K} = \mathbb{R}$. Let $X_{\mathbb{C}} = X \oplus iX$ and $T_{\mathbb{C}}(u + iv) = Tu + iTv$ be complexifications of X and T respectively. It is straightforward to see that $T_{\mathbb{C}}$ satisfies the same conditions with $E_{\mathbb{C}} = E + iE$ and $S_{\mathbb{C}}(u + iv) = Su + iSv$ taken as E and S. Corollary 1.10 in the complex case implies that $H_0 = \mathfrak{H}\{zT_{\mathbb{C}}\colon\ z \in \mathbb{C},\ b^{-1} < |z| < a^{-1}\}$ is a dense G_δ-subset of $X_{\mathbb{C}}$. Clearly $H = \mathfrak{H}\{zT\colon\ z \in \mathbb{R},\ b^{-1} < |z| < a^{-1}\}$ contains the projection of H_0 onto X along iX and therefore in dense in X. The fact that H is a G_δ-subset of X follows from Corollary 2.3.

5. Counterexamples on hypercyclic scalar multiples

We find operators, whose existence is assured by Theorem 1.11 in the class of bilateral weighted shifts on $\ell_2(\mathbb{Z})$. Recall that if $w = \{w_n\}_{n\in\mathbb{Z}}$ is a bounded sequence of non-zero scalars, then the unique $T_w \in L(\ell_2(\mathbb{Z}))$ such that $T_w e_n = w_n e_{n-1}$ for $n \in \mathbb{Z}$, where $\{e_n\}_{n\in\mathbb{Z}}$ is the canonical orthonormal basis of the Hilbert space $\ell_2(\mathbb{Z})$, is called the *bilateral weighted shift with the weight sequence* w. Hypercyclicity of bilateral weighted shifts was characterized by Salas [20], whose necessary and sufficient condition is presented in a more convenient shape in [22].

Theorem S. *Let T_w be a bilateral weighted shift on $\ell_2(\mathbb{Z})$. Then T_w is hypercyclic if and only if for any $k \in \mathbb{Z}_+$,*

$$\varliminf_{n\to\infty} \bigl(\widetilde{w}(k - n + 1, k) + \widetilde{w}(k + 1, k + n)^{-1}\bigr) = 0,$$

$$\text{where } \widetilde{w}(a, b) = \prod_{j=a}^{b} |w_j| \text{ for } a, b \in \mathbb{Z},\ a \leqslant b. \tag{5.1}$$

It is well known and easy to see that a bilateral weighted shift T_w is invertible if and only if $\inf_{n\in\mathbb{Z}}|w_n|>0$. In this case condition (5.1) can be rewritten in the following simpler form.

Theorem S′. *Let T_w be an invertible bilateral weighted shift on $\ell_2(\mathbb{Z})$. Then T_w is hypercyclic if and only if*

$$\varliminf_{n\to\infty}\left(\widetilde{w}(-n,0)+\widetilde{w}(0,n)^{-1}\right)=0. \tag{5.2}$$

5.1. Proof of Theorem 1.11, Part II

First, we prove few elementary lemmas. The following one generalizes the fact that the set of hypercyclic vectors of a hypercyclic operator is dense.

Lemma 5.1. *Let X be a topological vector space and $\mathcal{A}$ be a family of pairwise commuting continuous linear operators on X. Then the set $\mathfrak{H}(\mathcal{A})=\bigcap_{T\in\mathcal{A}}\mathfrak{H}(T)$ is either empty or dense in X.*

Proof. Let $x\in\mathfrak{H}(\mathcal{A})$ and $S\in\mathcal{A}$. We have to show that $\mathfrak{H}(\mathcal{A})$ is dense in X. Since x is a hypercyclic vector for S, $O(S,x)=\{S^nx\colon n\in\mathbb{Z}_+\}$ is dense in X and therefore S has dense range. Take any $T\in\mathcal{A}$. Since $TS=ST$, $O(T,S^mx)=S^m(O(T,x))$ for each $m\in\mathbb{Z}_+$. Since $x\in\mathfrak{H}(T)$ and S^m has dense range, $O(T,S^mx)$ is dense in X. Hence $S^mx\in\mathfrak{H}(T)$ for any $T\in\mathcal{A}$ and $m\in\mathbb{Z}_+$. That is, $O(S,x)\subseteq\mathfrak{H}(\mathcal{A})$. Since $O(S,x)$ is dense in X, so is $\mathfrak{H}(\mathcal{A})$. □

Lemma 5.2. *Let X be a locally convex topological vector space, $T\in L(X)$, $A\subseteq(0,\infty)$ and $x\in\mathfrak{H}\{cT\colon c\in A\}$. Assume also that there exists a non-empty open subset U of X such that*

$$\sum_{n\in Q_U}n^{-1}<\infty,\quad \textit{where } Q_U=\left\{n\in\mathbb{N}\colon a^nT^nx\in U \textit{ for some } a\in A\right\}. \tag{5.3}$$

Then A has zero Lebesgue measure.

Proof. Clearly we can assume that $A\neq\varnothing$ and therefore $\Lambda\neq\varnothing$, where $\Lambda=\ln(A)=\{\ln a\colon a\in A\}$. Since X is Hausdorff and locally convex, we can find a continuous seminorm p on X such that $V=U\cap\{u\in X\colon 1<p(u)<e\}$ is non-empty. It suffices to show that Λ has zero Lebesgue measure. Let $\alpha\in\Lambda$ and $m\in\mathbb{N}$. Since x is hypercyclic for $e^\alpha T$ and V is open, we can find $n\geqslant m$ such that $e^{\alpha n}T^n\in V\subseteq U$. Then $n\in Q_U$ and $p(e^{\alpha n}T^nx)\in(1,e)$. Hence

$$\alpha\in(\alpha_n,\beta_n),\quad \text{where } \alpha_n=\frac{-\ln(p(T^nx))}{n} \text{ and } \beta_n=\frac{1-\ln(p(T^nx))}{n}.$$

Since $\alpha\in\Lambda$ is arbitrary, we obtain

$$\Lambda\subseteq\bigcup_{n\in Q_U,\,n\geqslant m}(\alpha_n,\beta_n)\quad\text{for any } m\in\mathbb{N}.$$

On the other hand, (α_n, β_n) is an interval of length n^{-1}. Then (5.3) and the last display imply that Λ can be covered by intervals with arbitrarily small sum of lengths. That is, Λ has zero Lebesgue measure. $\square$

For $k \in \mathbb{N}$, we denote

$$m_k = 2^{3k^2}, \qquad I_k^- = \left\{n \in \mathbb{N}\colon \frac{7}{8}m_k \leqslant n < m_k\right\}, \qquad I_k^+ = \left\{n \in \mathbb{N}\colon m_k < n \leqslant \frac{9}{8}m_k\right\} \quad \text{and}$$

$$I_k = I_k^- \cup I_k^+ \cup \{m_k\} = \left\{n \in \mathbb{N}\colon \frac{7}{8}m_k \leqslant n \leqslant \frac{9}{8}m_k\right\}. \tag{5.4}$$

Consider the sequence $w = \{w_n\}_{n\in\mathbb{Z}}$ defined by the formula

$$w_n = \begin{cases} 2^8 & \text{if } n \in I_k^- \cup -I_k^+,\ k \in \mathbb{N}, \\ 2^{-8} & \text{if } n \in I_k^+ \cup -I_k^-,\ k \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases} \tag{5.5}$$

Clearly w is a sequence of positive numbers and $0 < 2^{-8} = \inf_{n\in\mathbb{Z}} w_n < \sup_{n\in\mathbb{Z}} w_n = 2^8 < \infty$. Hence T_w is an invertible bilateral weighted shift. In order to prove Part II of Theorem 1.11 it is enough to verify the following statement.

Example 5.3. Let w be the weight sequence defined by (5.5) and $T = T_w$ be the corresponding bilateral weighted shift on $\ell_2(\mathbb{Z})$. Then $M_T = (1/2, 2)$ and any $\Lambda \subseteq (1/2, 2)$ has Lebesgue measure 0 if the family $\{aT\colon a \in \Lambda\}$ has a common hypercyclic vector.

Proof. Using the definition (5.5) of the sequence w, it is easy to verify that for any $n \in \mathbb{N}$,

$$\beta(n) = \begin{cases} 2^{8n-7m_k+8} & \text{if } n \in I_k^-,\ k \in \mathbb{N}, \\ 2^{9m_k-8n} & \text{if } n \in I_k^+,\ k \in \mathbb{N}, \\ 1 & \text{otherwise,} \end{cases} \quad \text{where } \beta(n) = \prod_{j=0}^{n} w_j. \tag{5.6}$$

Moreover, $w_n^{-1} = w_{-n}$ for any $n \in \mathbb{Z}$. Using this fact and the equality $w_0 = 1$, we get

$$\widetilde{w}(j,n) = \begin{cases} \beta(n)\beta(j-1)^{-1} & \text{if } j \geqslant 1, \\ \beta(-1-n)\beta(-j)^{-1} & \text{if } n \leqslant -1, \\ \beta(n)\beta(-j)^{-1} & \text{if } j \leqslant 0 \text{ and } n \geqslant 0 \end{cases} \quad \text{for any } j, n \in \mathbb{Z},\ j \leqslant n, \tag{5.7}$$

where the numbers $\widetilde{w}(j,n)$ are defined in (5.1). In particular, $\widetilde{w}(0,n) = \beta(n)$ and $\widetilde{w}(-n,0) = \beta(n)^{-1}$ for each $n \in \mathbb{N}$. This observation together with Theorem S$'$ and the fact that $aT = T_{aw}$ for $a \neq 0$ imply that for $a > 0$,

$$aT \text{ is hypercyclic if and only if } \varliminf_{n\to\infty} \beta(n)^{-1}\left(a^n + a^{-n}\right) = 0. \tag{5.8}$$

By (5.6), $1 \leqslant \beta(n) \leqslant 2^n$ for $n \in \mathbb{N}$, which together with (5.8) implies that $M_T \subseteq (1/2, 2)$. On the other hand, by (5.6), $\beta(m_k) = 2^{m_k}$ for each $k \in \mathbb{N}$. Hence $\beta(m_k)^{-1}(a^{m_k} + a^{-m_k}) \to 0$ as $k \to \infty$ for any $a \in (1/2, 2)$. According to (5.8), aT is hypercyclic if $1/2 < a < 2$. Hence $M_T = (1/2, 2)$.

Let now Λ be a non-empty subset of $(1/2, 2)$ such that the family $\{aT\colon a \in \Lambda\}$ has common hypercyclic vectors. We have to demonstrate that Λ has zero Lebesgue measure. Pick $\varepsilon > 0$ such that $\frac{\varepsilon}{1-\varepsilon} < 2^{-8}$. By Lemma 5.1, there is a common hypercyclic vector x of the family $\{aT\colon a \in \Lambda\}$ such that $\|x - e_{-1}\| < \varepsilon$. Let

$$Q = \{n \in \mathbb{N}\colon \|a^n T^n x - e_0\| < \varepsilon \text{ for some } a \in \Lambda\} \quad \text{and} \quad J = \bigcup_{k=1}^{\infty} I_k.$$

First, we show that $Q \subseteq J$. Let $n \in Q$. Then there is $a \in \Lambda$ such that $\|a^n T^n x - e_0\| < \varepsilon$. Hence

$$|\langle a^n T^n x, e_0\rangle| > 1 - \varepsilon \quad \text{and} \quad |\langle a^n T^n x, e_{-n-1}\rangle| < \varepsilon.$$

Using (5.7), we get $\langle a^n T^n x, e_0\rangle = a^n \beta(n) x_n$ and $\langle a^n T^n x, e_{-n-1}\rangle = a^n \beta(n)^{-1} x_{-1}$. Then from the last display it follows that

$$a^n \beta(n)|x_n| > 1 - \varepsilon \quad \text{and} \quad a^n \beta(n)^{-1} w_n |x_{-1}| < \varepsilon.$$

Since $\|x - e_{-1}\| < \varepsilon$, $|x_{-1}| > 1 - \varepsilon$ and $|x_n| < \varepsilon$. Then according to the last display,

$$\beta(n) > \frac{1-\varepsilon}{\varepsilon} \max\{a^n, a^{-n}\} \geqslant \frac{1-\varepsilon}{\varepsilon} > 2^8 > 1.$$

By (5.6), $\beta(j) = 1$ if $j \notin J$. Hence $n \in J$. Since n is an arbitrary element of Q, we get $Q \subseteq J$.

Next, we show that $(Q - Q) \cap \mathbb{N} \subseteq J$. Indeed, let $m, n \in Q$ be such that $m > n$. Since $m, n \in Q$, we can pick $a, b \in \Lambda$ such that $\|a^n T^n x - e_0\| < \varepsilon$ and $\|b^m T^m x - e_0\| < \varepsilon$. In particular,

$$\begin{aligned} &|\langle a^n T^n x, e_0\rangle| > 1 - \varepsilon, && |\langle b^m T^m x, e_0\rangle| > 1 - \varepsilon, \\ &|\langle a^n T^n x, e_{m-n}\rangle| < \varepsilon \quad \text{and} && |\langle b^m T^m x, e_{n-m}\rangle| < \varepsilon. \end{aligned}$$

Using (5.7), we get

$$\begin{aligned} \langle a^n T^n x, e_0\rangle &= a^n \beta(n) x_n, & \langle a^n T^n x, e_{m-n}\rangle &= a^n \beta(m)\beta(m-n)^{-1} x_m, \\ \langle b^m T^m x, e_0\rangle &= b^m \beta(m) x_m, & \langle b^m T^m x, e_{n-m}\rangle &= b^m \beta(n)\beta(m-n-1)^{-1} x_n. \end{aligned}$$

According to the last two displays,

$$\beta(m-n-1) > \frac{1-\varepsilon}{\varepsilon} a^n b^{-m} \quad \text{and} \quad \beta(m-n) > \frac{1-\varepsilon}{\varepsilon} a^{-n} b^m.$$

Since $\beta(m-n) = \beta(m-n-1) w_{m-n} \geqslant 2^{-8} \beta(m-n-1)$ from the last display it follows that

$$\beta(m-n) > 2^{-8} \frac{1-\varepsilon}{\varepsilon} \max\{a^n b^{-m}, a^{-n} b^m\} \geqslant 2^{-8} \frac{1-\varepsilon}{\varepsilon} > 1.$$

Since $\beta(j) = 1$ if $j \notin J$, we have $m - n \in J$. Hence $(Q - Q) \cap \mathbb{N} \subseteq J$.

Let now $k \in \mathbb{N}$ and $m, n \in Q \cap I_k$ be such that $m > n$. Since $(Q - Q) \cap \mathbb{N} \subseteq J$, we have $m - n \in J$. Since $m, n \in I_k$, we get $m - n \leqslant \frac{m_k}{4} < \frac{7m_k}{8} = \min I_k$. Hence $m - n \in \bigcup_{j=0}^{k-1} I_j$, where $I_0 = \varnothing$. Then $|m - n| \leqslant \frac{9m_{k-1}}{8} < 2m_{k-1}$, where $m_0 = 1$. Hence $Q \cap I_k$ has at most $2m_{k-1}$ elements. On the other hand, $n \geqslant \frac{7m_k}{8} \geqslant \frac{m_k}{2}$ for any $n \in I_k$ and therefore

$$\sum_{n \in Q \cap I_k} n^{-1} \leqslant 2m_{k-1} \frac{2}{m_k} = \frac{4m_{k-1}}{m_k} \leqslant 2^{-k},$$

where the last inequality follows from the definition of m_k. Since $Q \subseteq J$ and J is the union of disjoint sets I_k, we obtain

$$\sum_{n \in Q} n^{-1} = \sum_{k=1}^{\infty} \sum_{n \in Q \cap I_k} n^{-1} \leqslant \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty.$$

Using the definition of Q and Lemma 5.2, we now see that Λ has zero Lebesgue measure. $\square$

5.2. Proof of Theorem 1.11, Part I

Consider the sequences $\{a_n\}_{n \in \mathbb{Z}}$ and $\{w_n\}_{n \in \mathbb{Z}}$ defined by the formulae

$$a_n = \begin{cases} 1 & \text{if } |n| \leqslant 5 \text{ or } -2 \cdot 5^k \leqslant n < -5^k, \\ & \text{or } -5^{k+1} \leqslant n < -4 \cdot 5^k,\ k \in \mathbb{N}, \\ 8^{-1} & \text{if } -3 \cdot 5^k \leqslant n < -2 \cdot 5^k,\ k \in \mathbb{N}, \\ 8 & \text{if } -4 \cdot 5^k \leqslant n < -3 \cdot 5^k,\ k \in \mathbb{N}, \\ 2^{-1} & \text{if } 2 \cdot 5^k < n \leqslant 4 \cdot 5^k,\ k \in \mathbb{N}, \\ 4^{-1} & \text{if } 5^k < n \leqslant 2 \cdot 5^k,\ k \in \mathbb{N}, \\ 16 & \text{if } 4 \cdot 5^k < n \leqslant 5^{k+1},\ k \in \mathbb{N}; \end{cases} \qquad w_n = \begin{cases} 1 & \text{if } |n| \leqslant 1, \\ n(n-1)^{-1} a_n & \text{if } n \geqslant 2, \\ (n+1)n^{-1} a_n & \text{if } n \leqslant -2. \end{cases} \tag{5.9}$$

It is easy to see that w is a bounded sequence of positive numbers and $\inf_{n \in \mathbb{Z}} w_n > 0$. Hence the bilateral weighted shift T_w is invertible. In order to prove Part I of Theorem 1.11 it is enough to verify the following statement.

Example 5.4. Let w be the weight sequence defined by (5.9) and $S = T_w$ be the corresponding bilateral weighted shift on $\ell_2(\mathbb{Z})$. Then $M_S = \{1, 2\}$.

Proof. Using (5.9), one can easily verify that

$$\gamma_+(n) = \begin{cases} 4^{5^k - n} & \text{if } 5^k < n \leqslant 2 \cdot 5^k,\ k \in \mathbb{N}, \\ 2^{-n} & \text{if } 2 \cdot 5^k < n \leqslant 4 \cdot 5^k,\ k \in \mathbb{N}, \\ 16^{n - 5^{k+1}} & \text{if } 4 \cdot 5^k < n \leqslant 5^{k+1},\ k \in \mathbb{N}, \end{cases} \quad \text{where } \gamma_+(n) = \prod_{j=0}^{n} a_j, \tag{5.10}$$

$$\gamma_-(n) = \begin{cases} 1 & \text{if } 5^k < n \leqslant 2 \cdot 5^k \text{ or } 4 \cdot 5^k < n \leqslant 5^{k+1},\ k \in \mathbb{N}, \\ 8^{2 \cdot 5^k - n} & \text{if } 2 \cdot 5^k < n \leqslant 3 \cdot 5^k,\ k \in \mathbb{N}, \\ 8^{n - 4 \cdot 5^k} & \text{if } 3 \cdot 5^k < n \leqslant 4 \cdot 5^k,\ k \in \mathbb{N}, \end{cases} \quad \text{where } \gamma_-(n) = \prod_{j=-n}^{0} a_j. \tag{5.11}$$

For brevity we denote $\beta_+(n)=\widetilde{w}(0,n)$ and $\beta_-(n)=\widetilde{w}(-n,0)$, where $\widetilde{w}(k,l)$ are defined in (5.1). By definition of w,

$$\beta_+(n)=n\gamma_+(n)\quad\text{and}\quad\beta_-(n)=\frac{\gamma_-(n)}{n}\quad\text{for any } n\in\mathbb{N}. \tag{5.12}$$

According to (5.10) and (5.11), $\gamma_+(5^k)=\gamma_-(5^k)=1$ and $\gamma_+(3\cdot 5^k)=\gamma_-(3\cdot 5^k)=8^{-5^k}$ for any $k\in\mathbb{N}$. Using (5.12), we get $\beta_+(5^k)^{-1}=\beta_-(5^k)=5^{-k}\to 0$ and $(2^{3\cdot 5^k}\beta_+(3\cdot 5^k))^{-1}=2^{3\cdot 5^k}\beta_-(3\cdot 5^k)=3^{-1}5^{-k}\to 0$ as $k\to\infty$. Applying Theorem S′ to $S=T_w$ and $2S=T_{2w}$, we see that S and $2S$ are both hypercyclic.

Let $c>0$ be such that $cS=T_{cw}$ is hypercyclic. By Theorem S′, there exists a strictly increasing sequence $\{n_j\}_{j\in\mathbb{N}}$ of positive integers such that

$$\bigl(c^{n_j}\beta_+(n_j)\bigr)^{-1}+c^{n_j}\beta_-(n_j)\to 0\quad\text{as } j\to\infty. \tag{5.13}$$

Let k_j be the integer part of $\log_5 n_j$. Then $n_j=b_j5^{k_j}$, where $1\leqslant b_j<5$. Passing to a subsequence, if necessary, we can additionally assume that $b_j\to b\in[1,5]$ as $j\to\infty$. Using (5.10) and (5.11), one can easily verify that convergence of b_j to b implies that

$$\lim_{j\to\infty}\gamma_+(n_j)^{1/n_j}=\lambda_+(b)\quad\text{and}\quad\lim_{j\to\infty}\gamma_-(n_j)^{1/n_j}=\lambda_-(b), \tag{5.14}$$

where the continuous positive functions λ_+ and λ_- on $[1,5]$ are defined by the formula

$$\lambda_+(b)=\begin{cases}4^{b^{-1}-1} & \text{if } 1\leqslant b<2,\\ 1/2 & \text{if } 2\leqslant b\leqslant 4,\\ 16^{1-5b^{-1}} & \text{if } 4<b\leqslant 5\end{cases}\quad\text{and}\quad\lambda_-(b)=\begin{cases}1 & \text{if } b\in[1,2]\cup[4,5],\\ 8^{2b^{-1}-1} & \text{if } 2<b\leqslant 3,\\ 8^{1-4b^{-1}} & \text{if } 3<b<4.\end{cases} \tag{5.15}$$

According to (5.12),

$$\lim_{n\to\infty}\left(\frac{\beta_+(n)}{\gamma_+(n)}\right)^{1/n}=1\quad\text{and}\quad\lim_{n\to\infty}\left(\frac{\beta_-(n)}{\gamma_-(n)}\right)^{1/n}=1.$$

From (5.14) and the above display it follows that

$$\lim_{j\to\infty}\bigl(c^{n_j}\beta_+(n_j)^{1/n_j}\bigr)^{-1/n_j}=\bigl(c\lambda_+(b)\bigr)^{-1}\quad\text{and}\quad\lim_{j\to\infty}\bigl(c^{n_j}\beta_+(n_j)^{1/n_j}\bigr)^{1/n_j}=c\lambda_-(b).$$

These equalities together with (5.13) imply that $(c\lambda_+(b))^{-1}\leqslant 1$ and $c\lambda_-(b)\leqslant 1$. In particular, $\frac{\lambda_-(b)}{\lambda_+(b)}\leqslant 1$. On the other hand, (5.15) implies that $\frac{\lambda_-(b)}{\lambda_+(b)}>1$ for $b\in(1,3)\cup(3,5)$. Hence $b\in\{1,3,5\}$. If $b\in\{1,5\}$, then $\lambda_-(b)=\lambda_+(b)=1$ and the inequalities $(c\lambda_+(b))^{-1}\leqslant 1$ and $c\lambda_-(b)\leqslant 1$ imply that $c\leqslant 1$ and $c^{-1}\leqslant 1$. That is, $c=1$. If $b=3$, then $\lambda_-(b)=\lambda_+(b)=1/2$ and the inequalities $(c\lambda_+(b))^{-1}\leqslant 1$ and $c\lambda_-(b)\leqslant 1$ imply that $c/2\leqslant 1$ and $2/c\leqslant 1$. That is, $c=2$. Thus $c\in\{1,2\}$. Hence $M_S=\{1,2\}$. $\square$

6. Proof of Theorem 1.3

The main tool in the proof is the following result by Macintyre and Fuchs. The following theorem is a part of Theorem 1 in [19].

Theorem MF. *Let $d > 0$, $n \in \mathbb{N}$ and $z_1, \dots, z_n \in \mathbb{C}$. Then there exist n closed disks $D_1, \dots, D_n$ on the complex plane such that their radii $r_1, \dots, r_n$ satisfy $\sum_{j=1}^n r_j^2 \leqslant 4d^2$ and*

$$\sum_{j=1}^{n} |z - z_j|^{-2} < \frac{n(1 + \ln n)}{d^2} \quad \textit{for any } z \in \mathbb{C} \setminus \bigcup_{j=1}^{n} D_j. \tag{6.1}$$

We also need the following elementary lemma.

Lemma 6.1. *Let X be a topological vector space, $T \in L(X)$ and $f \in X^* \setminus \{0\}$. Assume also that there exists a polynomial p such that $p(T)$ is hypercyclic. Then the sequence $\{(T^*)^n f\}_{n\in\mathbb{Z}_+}$ is linearly independent.*

Proof. Assume that the sequence $\{(T^*)^n f\}_{n\in\mathbb{Z}_+}$ is linearly dependent. Then we can pick $n \in \mathbb{N}$ such that $(T^*)^n f \in L = \mathtt{span}\{f, T^* f, \dots, (T^*)^{n-1} f\}$. It follows that L is a non-trivial finite dimensional invariant subspace for T^*. Hence $L^\perp = \{x \in X\colon\ g(x) = 0 \text{ for any } g \in L\}$ is a closed linear subspace of X of finite positive codimension invariant for T. Clearly $L^\perp$ is also invariant for $p(T)$. We have obtained a contradiction with a result of Wengenroth [24], according to which hypercyclic operators on topological vector spaces have no closed invariant subspaces of positive finite codimension. □

We are ready to prove Theorem 1.3. Let X be a complex topological vector space such that $X^* \neq \{0\}$, $T \in L(X)$ and Λ be a non-empty subset of $\mathbb{R} \times \mathbb{C}$ for which the family $\mathcal{A} = \{e^a(T + bI)\colon\ (a, b) \in \Lambda\}$ has a common hypercyclic vector. In order to prove Theorem 1.3 it suffices to show that Λ has zero three-dimensional Lebesgue measure. Pick a non-zero $f \in X^*$. By Lemma 5.1, the set $\mathfrak{H}(\mathcal{A})$ of common hypercyclic vectors for operators from $\mathcal{A}$ is dense in X. Since $\mathfrak{H}(\mathcal{A})$ is also closed under multiplications by non-zero scalars, we can pick $x \in \mathfrak{H}(\mathcal{A})$ such that $f(x) = 1$. For each $n \in \mathbb{N}$ consider the complex polynomial

$$p_n(b) = f\big((T + bI)^n x\big) = \sum_{j=0}^{n} \binom{n}{j} \big(\big(T^*\big)^{n-j} f\big)(x) b^j. \tag{6.2}$$

Clearly p_n is a polynomial of degree n with coefficient $1 = f(x)$ in front of b^n (such polynomials are usually called *monic*). Differentiating (6.2) by b, we obtain that $p_n'(b) = nf((T + bI)^{n-1}x) = np_{n-1}(b)$. That is,

$$p_n' = np_{n-1} \quad \text{for each } n \in \mathbb{N}. \tag{6.3}$$

Applying (6.3) twice, one can easily verify that

$$\big(p_n'/p_n\big)' = n^2\left(\left(1 - \frac{1}{n}\right)\frac{p_{n-2}}{p_n} - \left(\frac{p_{n-1}}{p_n}\right)^2\right) \quad \text{for each } n \geqslant 2. \tag{6.4}$$

The equality (6.4) immediately implies the following inequality:

$$\left|\left(p_n'/p_n\right)'\right| \geqslant n^2\left(\left|\frac{p_{n-2}}{2p_n}\right| - \left|\frac{p_{n-1}}{p_n}\right|^2\right) \quad \text{for each } n \geqslant 2. \tag{6.5}$$

Lemma 6.2. *For any $(a,b) \in \Lambda$ and $k \in \mathbb{Z}_+$, the sequence $\{v_n\}_{n\geqslant k}$ is dense in $\mathbb{C}^{k+1}$, where $v_n = e^{an}(p_n(b), p_{n-1}(b), \dots, p_{n-k}(b))$.*

Proof. Assume the contrary. Then there exist $(a,b) \in \Lambda$ and a non-empty open subset W of $\mathbb{C}^{k+1}$ such that $v_n \notin W$ for each $n \geqslant k$. Let $S = e^a(T + bI)$. By definition of p_m, for $0 \leqslant j \leqslant k$,

$$e^{an} p_{n-j}(b) = e^{an} f\left((T+bI)^{n-j}x\right) = e^{aj} f\left(S^{n-j}x\right) = e^{aj}\left(S^*\right)^{k-j} f\left(S^{n-k}x\right).$$

Thus the relation $v_n \notin W$ can be rewritten as $S^{n-k}x \notin R^{-1}(W)$, where the linear operator $R: X \to \mathbb{C}^{k+1}$ is defined by the formula

$$(Ry)_l = e^{a(l-1)}\left(S^*\right)^{k-l+1} f(y) \quad \text{for } 1 \leqslant l \leqslant k+1.$$

By Lemma 6.1, continuous linear functionals $f, S^*f, \dots, (S^*)^k f$ are linearly independent. It follows that R is continuous and surjective. Hence $V = R^{-1}(W)$ is a non-empty open subset of X. Thus $S^{n-k}x$ does not meet the non-empty open set V for each $n \geqslant k$, which is impossible since $x \in \mathfrak{H}(S)$. $\square$

By Lemma 6.2 with $k = 2$, for any $(a,b) \in \Lambda$, the sequence $\{v_n = e^{an}(p_n(b), p_{n-1}(b), p_{n-2}(b))\}_{n\geqslant 2}$ is dense in $\mathbb{C}^3$. Since the map $F: \mathbb{C}^\star \times \mathbb{C}^2 \to \mathbb{C}^3$, $F(u,v,w) = (u, v/u, w/u)$ is continuous and has dense range, $\{F(u_n): n \geqslant 2,\ p_n(b) \neq 0\}$ is dense in $\mathbb{C}^3$. That is,

$$\left\{\left(e^{an}p_n(b), p_{n-1}(b)/p_n(b), p_{n-2}(b)/p_n(b)\right): n \geqslant 2,\ p_n(b) \neq 0\right\} \text{ is dense in } \mathbb{C}^3.$$

It follows that any $(a,b) \in \Lambda$ is contained in infinitely many sets C_n, where

$$C_n = \left\{(a,b) \in \mathbb{R}\times\mathbb{C}:\ 1 < \left|e^{an}p_n(b)\right| < e,\ \left|p_{n-1}(b)/p_n(b)\right| < 1,\ \left|p_{n-2}(b)/p_n(b)\right| > 8\right\}.$$

That is,

$$\Lambda \subseteq \Lambda^* = \bigcap_{m=1}^{\infty} \bigcup_{n\geqslant m} C_n. \tag{6.6}$$

Clearly, $C_n \subseteq \mathbb{R} \times B_n$, where

$$B_n = \left\{b \in \mathbb{C}:\ \left|p_{n-1}(b)/p_n(b)\right| < 1,\ \left|p_{n-2}(b)/p_n(b)\right| > 8\right\}.$$

Applying the inequality (6.5), we see that

$$B_n \subseteq B_n' = \left\{b \in \mathbb{C}:\ \left|\left(p_n'(b)/p_n(b)\right)'\right| \geqslant 3n^2\right\}. \tag{6.7}$$

Since p_n is a monic polynomial of degree n, there exist $z_1, \dots, z_n \in \mathbb{C}$ such that

$$p_n(b) = \prod_{j=1}^{n} (b - z_j) \quad \text{and therefore} \quad \bigl(p_n'(b)/p_n(b)\bigr)' = -\sum_{j=1}^{n} (b - z_j)^{-2}.$$

By Theorem MF with $d = n^{-1/3}$, there are n closed disks $D_1, \dots, D_n$ on the complex plane such that their radii $r_1, \dots, r_n$ satisfy

$$\sum_{j=1}^{n} r_j^2 \leqslant 4n^{-2/3} \quad \text{and}$$

$$\bigl|\bigl(p_n'(b)/p_n(b)\bigr)'\bigr| \leqslant \sum_{j=1}^{n} |b - z_j|^{-2} < n^{5/3}(1 + \ln n) \quad \text{for any } b \in \mathbb{C} \setminus \bigcup_{j=1}^{n} D_j.$$

Since $n^{5/3}(1 + \ln n) \leqslant 3n^2$ for any $n \in \mathbb{N}$, we see that $B_n' \subseteq \bigcup_{j=1}^{n} D_j$. Hence

$$\mu_2(B_n) \leqslant \mu_2\bigl(B_n'\bigr) \leqslant \pi \sum_{j=1}^{n} r_j^2 \leqslant 4\pi n^{-2/3},$$

where μ_k is the k-dimensional Lebesgue measure. For each $b \in B_n$, $A_{b,n} = \{a \in \mathbb{R}\colon (a, b) \in C_n\}$ can be written as

$$A_{b,n} = \bigl\{a \in \mathbb{R}\colon 1 < \bigl|e^{an} p_n(b)\bigr| < e\bigr\} = \left(\frac{-\ln|p_n(b)|}{n}, \frac{1 - \ln|p_n(b)|}{n}\right),$$

which is an interval of length n^{-1}. Hence $\mu_1(A_{b,n}) = n^{-1}$ for each $b \in B_n$. By the Fubini theorem,

$$\mu_3(C_n) = \int_{B_n} \mu_1(A_{b,n}) \mu_2(db) = \frac{\mu_2(B_n)}{n} \leqslant 4\pi n^{-5/3}.$$

According to (6.6) and the above estimate, we obtain

$$\mu_3\bigl(\Lambda^*\bigr) \leqslant \inf_{m \in \mathbb{N}} 4\pi \sum_{n=m}^{\infty} n^{-5/3} = 0 \quad \text{since } \sum_{n=1}^{\infty} n^{-5/3} < \infty.$$

Thus $\mu_3(\Lambda^*) = 0$ and therefore $\mu_3(\Lambda) = 0$ since $\Lambda \subseteq \Lambda^*$. The proof of Theorem 1.3 is complete.

7. Concluding remarks and open problems

Lemma 6.1 implies the following easy corollary.

Corollary 7.1. *Let X be a topological vector space such that $0 < \mathtt{dim}\, X^* < \infty$. Then X supports no hypercyclic operators.*

Proof. Assume that $T \in L(X)$ is hypercyclic and $f \in X^*$, $f \neq 0$. By Lemma 6.1, the sequence $\{(T^*)^n f\}_{n\in\mathbb{Z}_+}$ is linearly independent, which contradicts the inequality $\dim X^* < \infty$. $\square$

In particular, $\mathcal{F}$-spaces $X = L_p[0,1] \times \mathbb{K}^n$ for $0 < p < 1$ and $n \in \mathbb{N}$ support no hypercyclic operators. Indeed, the dual of X is n-dimensional. On the other hand, each separable infinite dimensional Fréchet space supports a hypercyclic operator [6] and there are separable infinite dimensional $\mathcal{F}$-spaces [16] that support no continuous linear operators except the scalar multiples of I and therefore support no hypercyclic operators. However the following question remains open.

Question 7.2. *Let X be a separable $\mathcal{F}$-space such that X^* is infinite dimensional. Is it true that there exists a hypercyclic operator $T \in L(X)$?*

Part I of Theorem 1.11 shows that there exists a continuous linear operator S on ℓ_2 such that $M_S = \{1, 2\}$, where $M_S = \{a > 0\colon aS \text{ is hypercyclic}\}$. Using the same basic idea as in the proof of Theorem 1.11, one can construct an invertible bilateral weighted shift S on $\ell_2(\mathbb{Z})$ such that M_S is a dense subset of an interval and has zero Lebesgue measure. In particular, M_S and its complement are both dense in this interval. It is also easy to show that for any $\mathcal{F}$-space X and any $T \in L(X)$, M_T is a G_δ-set. If X is a Banach space, then M_T is separated from zero by the number $\|T\|^{-1}$. These observations naturally lead to the following question.

Question 7.3. *Characterize subsets A of $\mathbb{R}_+$ for which there is $S \in L(\ell_2)$ such that $A = M_S$. In particular, is it true that for any G_δ-subset A of $\mathbb{R}_+$ such that $\inf A > 0$, there exists $S \in L(\ell_2)$ for which $A = M_S$?*

In the proof of Part II of Theorem 1.11 we constructed an invertible bilateral weighted shift T on $\ell_2(\mathbb{Z})$ such that $M_T = (1/2, 2)$ and any subset A of $(1/2, 2)$ such that the family $\{aT\colon a \in A\}$ has a common hypercyclic vector must be of zero Lebesgue measure. It is also easy to see that our T enjoys the following extra property. Namely, if $E = \mathtt{span}\{e_n\colon n \in \mathbb{Z}\}$ and $x \in E$, then for $1/2 < \alpha < \beta < 2$, we have $\alpha^{-m_k} T^{m_k} x \to 0$ and $\beta^{m_k} T^{-m_k} x \to 0$ with $m_k = 2^{3k^2}$. This shows that the convergence to zero condition in Corollary 1.10 cannot be replaced by convergence to 0 of a subsequence. Note that, according to the hypercyclicity criterion [5], the latter still implies hypercyclicity of all relevant scalar multiples of T.

Recall that for $0 < s \leqslant 1$ the *Hausdorff outer measure* μ_s on $\mathbb{R}$ is defined as $\mu_s(A) = \lim_{\delta\downarrow 0} \mu_{s,\delta}(A)$ with $\mu_{s,\delta}(A) = \inf \sum (b_j - a_j)^s$, where the infimum is taken over all sequences $\{(a_j, b_j)\}$ of intervals of length $\leqslant \delta$, whose union contains A. The number $\inf\{s \in (0,1]\colon \mu_s(A) = 0\}$ is called the *Hausdorff dimension* of A. With basically the same proof Lemma 5.2 can be strengthened in the following way.

Lemma 7.4. *Let X be a locally convex topological vector space, $T \in L(X)$, $s \in (0,1]$, $A \subseteq (0,\infty)$ and x be a common hypercyclic vector for the family $\{cT\colon c \in A\}$. Assume also that there exists a non-empty open subset U of X such that $\sum_{n\in Q_U} n^{-s} < \infty$, where Q_U is defined in* (5.3). *Then $\mu_s(A) = 0$.*

Using Lemma 7.4 instead of Lemma 5.2, one can easily see that the operator T constructed in the proof of Part II of Theorem 1.11 has a stronger property. Namely, any $A \subset \mathbb{R}_+$ such that the family $\{cT\colon c \in A\}$ is hypercyclic has zero Hausdorff dimension.

Theorem CMP guarantees existence of common hypercyclic vectors for all non-identity operators of a universal strongly continuous semigroup $\{T_t\}_{t\geqslant 0}$ on an $\mathcal{F}$-space. On the other hand, Theorem CS shows that the non-identity elements of the 2-parametric translation group on $\mathcal{H}(\mathbb{C})$ have a common hypercyclic vector. The latter group enjoys the extra property of depending holomorphically on the parameter. Note that Theorem 1.2 strengthens this result.

Question 7.5. *Let X be a complex Fréchet space and $\{T_z\}_{z\in\mathbb{C}}$ be a holomorphic strongly continuous operator group. Assume also that for each $z\in\mathbb{C}^\star$, the operator T_z is hypercyclic. Is it true that the family $\{T_z\colon z\in\mathbb{C}^\star\}$ has a common hypercyclic vector?*

Question 7.6. *Let X be a complex Fréchet space and $\{T_z\}_{z\in\mathbb{C}}$ be a holomorphic strongly continuous operator group. Assume also that for each $z,a\in\mathbb{C}^\star$, the operator aT_z is hypercyclic. Is it true that the family $\{aT_z\colon a,z\in\mathbb{C}^\star\}$ has a common hypercyclic vector?*

An affirmative answer to the following question would allow to strengthen Theorem 1.7.

Question 7.7. *Let T be a continuous linear operator on a complex separable Fréchet space X and $0\leqslant a<b\leqslant\infty$. Assume also that for any $\alpha\in(a,b)$, the sets*

$$E_\alpha=\mathtt{span}\biggl(\bigcup_{|z|<\alpha}\mathtt{ker}(T-zI)\biggr)\quad\text{and}\quad F_\alpha=\mathtt{span}\biggl(\bigcup_{|z|>\alpha}\mathtt{ker}(T-zI)\biggr)$$

are both dense in X. Is it true that the family $\{zT\colon b^{-1}<|z|<a^{-1}\}$ has common hypercyclic vectors?

It is worth noting that according to the Kitai criterion for T from the above question, zT is hypercyclic for any $z\in\mathbb{C}$ with $b^{-1}<|z|<a^{-1}$. It also remains unclear whether the natural analog of Theorem 1.3 holds in the case $\mathbb{K}=\mathbb{R}$. For instance, the following question is open.

Question 7.8. *Does there exist a continuous linear operator T on a real Fréchet space such that the family $\{aT+bI\colon a>0,\ b\in\mathbb{R}\}$ has a common hypercyclic vector?*

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